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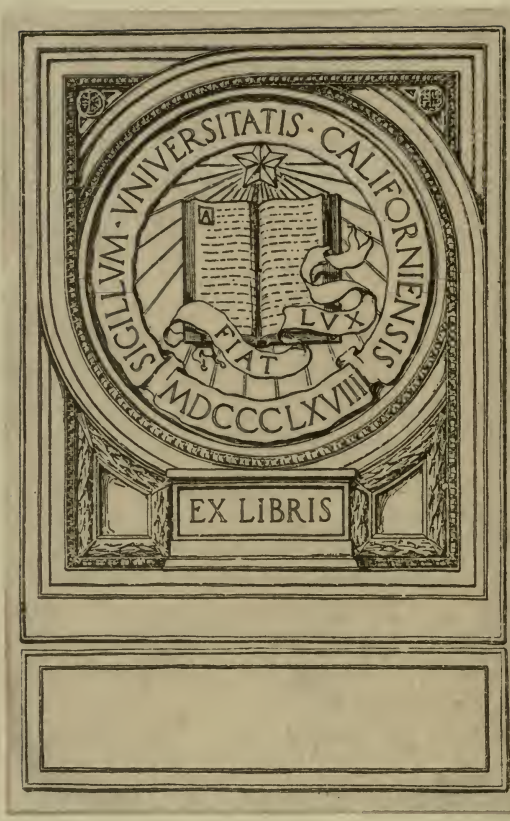
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MEMORANDA MATHEMATICA

A SYNOPSIS OF FACTS, FORMULAE, AND METHODS IN ELEMENTARY MATHEMATICS

BY

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PREFACE

Most students and teachers of Mathematics have at times wished for a book to which they might speedily refer for a particular formula or method. Such a want the author attempts to supply in the present volume, which collects in convenient compass the facts, formulae, and results of Elementary Mathematics and is intended, not to replace existing text-books, but to be a companion to them all, useful for revision and handy for reference.

The book embraces the subjects usually read in a school course with the exception of Arithmetic, which is only fragmentarily treated, and the more elementary parts of Geometry, which have less need of a summary than other subjects, since they are always presented in proposition form. The other branches of Elementary Mathematics are treated exhaustively, so that, as far as those subjects are concerned, the summary given should include almost all the requirements of a University course. It is hoped, therefore, that the book will be of special value to candidates for mathematical scholarships and to students in their first year who are reading for Mathematical Honours at the Universities.

Formulae with their deductions and applications are presented synoptically in accordance with a scheme enabling a reader to perceive analogies which he would possibly miss if dependent solely upon text-books. The

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methods by which the results are obtained are often indicated by a reference to other theorems upon which they depend, or by a summary of the proof, especially where the customary discussion calls for comment or modification. Notes are frequently added on matters of interest. Elaborate cross-references and a full index are provided; and, where subjects overlap, some repetition has been considered advisable.

The text has been set in two types. Matter is printed in larger type when, in the author's judgement, it should be committed to memory (but not, of course, verbatim). Matter in small type is confined to less important results, corollaries, and the solution of illustrative examples. Abbreviated proofs are also given in this form, together with data which are only occasionally required.

By arrangement with the author and publishers, the small but excellent collection of Mathematical Tables compiled by Mr. W. E. Paterson (Five-figure Logarithmic and Trigonometric Tables, Clarendon Press) has been appended to the book, and it is believed that this will increase its usefulness.

Every care has been taken in the correction of the proofs, but faults can scarcely be absent from a complicated work compiled in the intervals of a busy life, and the author will at all times be thankful to hear of any errata which may be detected, or to receive any suggestions for the improvement of the book.

BATH,
July, 1912.

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ARITHMETIC *

1. The Nine Remainder. Add up the digits of a number, rejecting all nines and all digits whose sum makes 9 and subtracting 9 whenever the sum is greater than 9. The single digit remainder obtained is the remainder after division of the number by 9.

Ex. 3782615. The work runs 5, 6, 12, [3], 5, 13, [4], 11, [2], 5. *Ans.* 5. The numbers in [ ] occur after subtracting a 9. It is best for the beginner to form the habit of beginning with the units digit.

The Eleven Remainder. Begin with units digit, add alternate digits, when the left of the number is reached return, subtracting from the sum obtained the digits which still remain. If when the right of the number is again reached the result is < 11 it is the remainder after dividing by 11. If it is greater than 11 subtract 11's until it becomes less. Elevens may be rejected or added at any stage.

Ex. 7 8 6 5 3 8 1. The work is 1, 4, 10, 17, (begin subtracting) 9, 4, (add 11) 15, 7. *Ans.* 7.

Proving by Nines and Elevens. To test a multiplication result, $MN = P$. Find the remainders m, n, p after dividing M, N, P by 9 or 11. Then the remainder of mn must be p.

Notes.—(1) The student is urged to test every completed multiplication result this way. The time required soon becomes infinitesimal.

(2) Instead of the Eleven Remainder the Nine Remainder may be used, but is not so satisfactory.

(3) A product which satisfies the test *may still be wrong.*

* The summary of this subject does not profess to be complete, and embraces only some points which are commonly forgotten. Enunciations are sometimes taken from the author's *Tutorial Arithmetic.*

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2. The Metric System.

Multiples (Greek prefixes): deca-, hecto-, kilo-.
Submultiples (Latin prefixes): deci-, centi-, milli-.

(1) *Length.* *Unit*: 1 metre = 39·37 inches.
Approximations: 1 centimetre = $\frac{2}{5}$ inch.
1 kilometre = $\frac{5}{8}$ mile.

(2) *Area.* *Unit*: 1 Are = 100 sq. metres.
Approximation: 1 Are = $\frac{1}{40}$ acre.

Note.—But scientific measurements are usually made in sq. cm. or sq. m.

(3) *Volume.* *Unit*: 1 litre = 1 cub. decimetre.
Approximation: 1 litre = $1\frac{3}{4}$ pints.

Note.—But scientific measurements are usually made in c. c.

(4) *Mass.* *Unit*: 1 gram = wt. of 1 c.c. of pure water at 4° C.
= 15·432 grains.

Note.—The accurate result is easier to remember than the approximation $15\frac{1}{2}$.

Approximation: 1 kilogram = $2\frac{1}{5}$ lb.

(5) *The C. G. S. system* of units is the system in which the centimetre, gram, second are the units respectively of length, mass, and time.

3. Terminating Decimals.

The following should be known by heart:

$\cdot 2 = \frac{1}{5}$ $\cdot 5 = \frac{1}{2}$
$\cdot 25 = \frac{1}{4}$ $\cdot 75 = \frac{3}{4}$
$\cdot 125 = \frac{1}{8}$ $\cdot 375 = \frac{3}{8}$ $\cdot 625 = \frac{5}{8}$ $\cdot 875 = \frac{7}{8}$.

Note.—These are of great use in simplifications, e. g. $\cdot 6875 = \cdot 6\frac{7}{8} = \frac{11}{16}$.

4. Recurring Decimals are subject to the following laws:

Law i. If N/D is a proper fraction in its lowest terms and D is a prime number, other than 2 or 5, then N/D

gives rise to a pure circulator and the number of places in the recurring period is either $D-1$ or one of the factors of $D-1$.

Note.—If $D \equiv \pm 7, \pm 11, \pm 17, \pm 19$ (mod. 40), N/D runs to the full period of $D-1$ figures or to an *odd* submultiple of that period. If $D \equiv \pm 1, \pm 3, \pm 9, \pm 13$, N/D runs to an *even* submultiple of $(D-1)$ figures. The last group of figures is easily remembered if desired: 1, 3, 9 and $1+3+9$.

Law ii. If D and E are different prime numbers other than 2 and 5 and $1/D$ has a period of p figures, while $1/E$ has a period of q figures, then N/DE (N prime to DE) has a period of r figures where r is the L. C. M. of p and q.

Note.—Similarly for three or more factors.

Law iii. If D is a prime number other than 2 and 5 and $1/D$ has a period of p figures, $1/D^2$ will *usually* have a period of pD figures, $1/D^3$ of pD^2, and so on.

Note.—The only exceptions at present known to this are $D = 3$ and $D = 487$; 1/487 and $1/487^2$ both have periods of 486 figures.

Law iv. If D is a number prime to 2 and 5 such that $1/D$ has a period of p figures, the fraction $1/2^m 5^n D$ is a mixed circulator with either m or n non-recurring figures (according as m or n is greater) and with p recurring figures.

Law v (*The Cyclic Law*). If D is a prime number other than 2 and 5 and $1/D$ has a period of p figures, then the fractions $1/D$, $2/D$, $3/D$ ... $(D-1)/D$ can be divided into $(D-1)/p$ groups, each group containing p fractions giving rise to a circulating period made up of the same figures, for all fractions of the group, taken in the same order round a circle, but starting at different points of the circle.

Ex. 1/13 circulates in 6 figures, ∴ there are $(13-1)/6 = 2$ groups. $\cdot\dot{0}76923$, $\cdot\dot{7}69230$, $\cdot\dot{6}92307$, $\cdot\dot{9}23076$, $\cdot\dot{2}30769$, $\cdot\dot{3}07692$ represent respectively $\frac{1}{13}$, $\frac{10}{13}$, $\frac{9}{13}$, $\frac{12}{13}$, $\frac{3}{13}$, and $\frac{4}{13}$, while the circulators formed from $\cdot\dot{1}53846 = \frac{2}{13}$ give the remaining 6 fractions with denominator 13.

Law vi (*The Complementary Law*). If N/D is a fraction in its lowest terms which gives rise to a circulator having an

even number of places in its recurring period (say $2n$), then the last n figures of the recurring period may be found from the first n figures by subtracting these successively from 9, provided D is *either* (1) a prime, *or* (2) the product of primes, none of which is capable separately of giving a period of n figures, *or* (3) the product of such primes with a power of 2 or 5 or both.

Ex. Thus in the Ex. of Law v, $076 + 923 = 999$, &c.

5. Recurring Decimals.

The following results should be learnt:

$\frac{1}{6} = \cdot 16\dot{}$;

$\frac{1}{7} = \cdot\dot{1}42857\dot{}$ with cyclical interchanges giving $\frac{2}{7}$, &c.;

$\frac{1}{13} = \cdot 0769230\dot{}$ with cyclical interchanges, &c. [§ 4, Law v, Ex.], and with easy deduction, mentally, of $\frac{2}{13}$.

Notes.—(1) As in § 3, $\cdot 8076923\dot{} = \cdot 8\frac{1}{13} = \frac{105}{130} = \frac{21}{26}$, &c.

(2) Many other results are not so much learnt as worked mentally when required, e.g. $\cdot\dot{3}$, $\cdot 8\dot{3}$, &c.

(3) To remember the periods for $\frac{1}{7}$ and $\frac{1}{13}$ note that $7 = 1+4+2$, and $13 = 0+7+6$. The Complementary Law gives the rest of the period.

6. Involution and Evolution.

(1) The following *squares* should be known by heart:

	1	2	3	4	5	6	7	8	9
0	1	4	9	16	25	36	49	64	81
1	121	144	169	196	225	256	289	324	361
2	441	484	529	576	625		729		

(2) The following *cubes* should be known by heart:

	1	2	3	4	5	6	7	8	9
0	1	8	27	64	125	216	343	512	729
1	1331	1728							

(3) The following *square roots* should be known :

$$\sqrt{2} = 1{\cdot}4142$$

$$\sqrt{3} = 1{\cdot}7320[5].$$

7. Logarithms. The following logarithms should be known :

$\log 2 = {\cdot}3010[300]$ $\log 5 = {\cdot}6990$

$\log 3 = {\cdot}4771[213]$ $\log 7 = {\cdot}8451$

$\log \pi = {\cdot}4971[5]$.

8. Numerical Data. The following numerical data are worth retaining. They are for the most part approximations only :

(1) A cubic foot of water weighs 1000 oz.

A pint of water weighs a pound and a quarter. (Exact.)

A gallon of water weighs 10 lb. (Exact.)

A gallon contains $277\frac{1}{4}$ cub. inches.

(2) A year is a leap-year when the last two figures of its number form a number divisible by 4, except for centuries, in which case the number of the century must be divisible by 4.

(3) The diameter of a halfpenny is 1 inch. Ten farthings, five halfpennies, or three pennies weigh 1 oz. Av. (Exact.)

(4) The radius of the Earth = 4000 miles = E, and its mean density is $5\frac{1}{2}$.

(5) The mean distance of the Moon = 60 E, its radius is $\frac{3}{11} E$, and its angular diameter = $\frac{1}{2}°$.

(6) The mean distance of the Sun = 93,000,000 miles, and its angular diameter = $\frac{1}{2}°$.

(7) The velocity of a shot from a big gun is from 1500 to 3000 ft. per second, the velocity of sound in air is 1100 ft. (332 m.) per second, and the velocity of light is 186,000 miles (300,000 Km.) per second.

MENSURATION

LENGTHS OF CURVED LINES

1. Circumference of Circle (radius r) $= 2\pi r$.

Values of π: 3·14159...

Approximations: 22/7, 355/113.

Note.—The approximation 355/113 easily remembered as 113)355(gives 6 decimal places, 3·141592... correct, and is almost as easy to work with as 22/7, which only gives 2 places.

2. Arc of Circle, subtending an $\angle\theta$ radians ($= A°$) at the centre, $\theta r = A\pi r/180$

$= \frac{1}{3}\{8 \text{ chord } \frac{1}{2} \text{ arc} - \text{chord arc}\}$ approximately.

Note.—If $A < 60°$, the percentage error of the approximate formula is ·1 or less.

3. Ellipse, Quadrant

$$= \frac{\pi a}{2}\left[1 - \left(\frac{e}{2}\right)^2 - \frac{3}{2!\,2!}\left(\frac{e}{2}\right)^4 - \frac{3^2.5}{3!\,3!}\left(\frac{e}{2}\right)^6 - \cdots\right].$$

Note.—$\frac{1}{4}\pi(a+b)$ gives the first two terms of this series and is a better approximation than $\frac{1}{2}\pi\sqrt{ab}$. If $e < \frac{1}{2}$ the error is $< ·1\%$.

4. Parabola (vertex A). If P be a point whose abscissa is x

$$\text{Arc } AP = \sqrt{ax+x^2} + a \log \frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}.$$

Areas of Plane Figures (Rectilinear Boundaries).

5. Triangle.

$\Delta = \frac{1}{2}$ base $\times \perp^{\text{r}}$ height,

$= \frac{1}{2}$ product of any two sides $\times$ sine of contained angle,

$= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

6. RECTANGLE. Area = length × breadth.

Square. Area = (side)^2.

Parallelogram. Area = any side × $\perp^r$ dist. to opp. side,
= $ab \sin\theta$. [a, b adjacent sides containing $\angle\theta$.]

7. TRAPEZIUM. Area = (mean of ∥ sides) ($\perp^r$ distance between them).

8. QUADRILATERAL. Divide into triangles.
[See also *TRIG.* § 14.]

9. REGULAR POLYGON. Divide into triangles from the centre. [*TRIG.* § 15.]

For reference.—The area of a regular polygon, side a, is ka^2, where k has the following values: Triangle (·4330), Square (1), Pentagon (1·7205), Hexagon (2·5981), Heptagon (3·6339), Octagon (4·8284), Nonagon (6·1818), Decagon (7·6942).

10. IRREGULAR POLYGON. If non-reentrant draw $\perp^{rs}$ from all angular points upon the longest side, thus dividing into 2 triangles and $n-3$ trapezia. If reentrant by triangles.

AREAS OF PLANE FIGURES (Curvilinear Boundaries).

11. CIRCLE. Area = πr^2.

Circular Ring. Area = $\pi(R^2 - r^2)$, where R is outer radius, r inner.
= $\pi(R-r)(R+r)$.

Sector of Circle subtending an $\angle A^\circ$ (= θ radians) at centre.

$$\text{Area} = \frac{A}{360} \times \pi r^2 = \tfrac{1}{2}\theta r^2.$$

Segment of Circle. Subtract Δ formed by radii and chord from sector.

12. ELLIPSE. Area = πab.

Segment of Ellipse.

$$\frac{\text{Area } APN}{\text{Area } AP'N} = \frac{b}{a}.$$

Sector of Ellipse.

Area ACP
= Areas $ANP + CNP$.

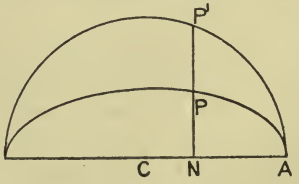


FIG. 1.

13. Parabola. The area cut off from a parabola by any chord $= \frac{2}{3}$ of that of a parallelogram with the chord as base and with the side parallel to the chord touching the parabola.

Note.—In particular if a double ordinate be drawn perpendicular to the major axis at a distance x from the vertex it cuts off from the curve an area $\frac{4}{3}xy = \frac{8}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}$, where $4a$ is the Latus Rectum.

14. Simpson's Approximate Rule for any curve. Area

$$pPZz = \tfrac{1}{3}\{y_1 + y_{2n+1} + 2(y_3 + y_5 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + \ldots + y_{2n})\}/h$$

$= \frac{1}{3}$[sum of first and last ordinates, twice the other odd ordinates, and four times all even ordinates] × distance between the ordinates.

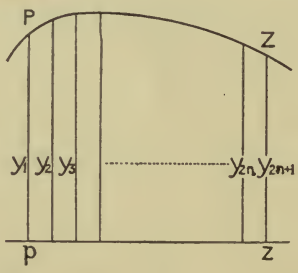


Fig. 2.

Notes.—(1) By far the best of the approximate formulae is the remarkable *Weddle's Rule*:

Divide the area by parallel and equidistant ordinates into groups of 6 spaces. For *each group* of 6 add together the odd ordinates (the first and the last are odd ordinates), six times the middle ordinate, and five times the two other even ordinates. Multiply the sum by three-tenths of the distance between the ordinates and the result is the area.

(2) If the ordinates are measured, and not calculated, the errors of Weddle's Rule are far inferior to the errors of measurement.

(3) Both Weddle's and Simpson's Rule give their best results when none of the ordinates are zero, and when they are numerous.

15. Similar Plane Figures. Areas proportional to the squares of corresponding sides.

Areas of Curved Surfaces.

16. Right Circular Cylinder.

Curved surface = perimeter of base × height
$= 2\pi rh.$

17. Right Circular Cone.

Curved surface = $\frac{1}{2}$ perimeter of base × slant height
$= \pi rl.$
$= \pi rh \cos\theta.$ ($h = \perp^{r}$ height, θ = semi-vertical angle.)
$= \pi r^2 \sin\theta.$

18. Sphere. Curved surface $= 4\pi r^2$.

Segment of Sphere.

Curved surface $= 2\pi rh.$ [$h = \perp^{r}$ height of segt.]

Belt of Sphere (cut off by ∥ planes).

Curved surface $= 2\pi rh.$ [$h = \perp^{r}$ dist. of planes.]

Note.—These results are particular cases of an important general theorem. *If a sphere just fits inside a cylinder two planes perpendicular to axis of the cylinder will cut off curved areas from the sphere and from the cylinder which are equal (and have the same centroid).*

Volumes of Solids.

19. Rectangular Parallelepiped.

Vol. = length × breadth × height
$= abc.$

Cube, edge a. Vol. $= a^3$.

20. Pyramid on any base: *Tetrahedron.*

Vol. = $\frac{1}{3}$ base × height.

21. Regular Solid, edge a. Volume $= ka^3$, where k has the following values: Tetrahedron ($\frac{1}{12}\sqrt{2}$), Cube (1), Octahedron ($\frac{1}{3}\sqrt{2}$), Dodecahedron $\frac{1}{4}(15+7\sqrt{5})$, Icosahedron $\frac{1}{12}(15+5\sqrt{5})$.

22. Prism. Vol. = cross-section × length.

23. *Frustum of Prism.* Vol. = cross-section × average length of edges.

24. Cylinder on any base (Right or Oblique).

Vol. = area of base × $\perp^{r}$ height.

25. Cone (Right or Oblique).

$$\text{Vol.} = \tfrac{1}{3} \text{ base} \times \perp^{\text{r}} \text{ height.}$$
$$= \tfrac{1}{3}\pi r^2 h \text{ (for rt. circular cone).}$$

26. *Frustum of Cone or Pyramid* [i. e. figure cut off by two planes ∥ to base]. If $A_1\, A_2$ be areas of ends of frustum and h the $\perp^{\text{r}}$ dist. between them,

$$\text{Vol.} = \tfrac{1}{3}h\left[A_1 + A_2 + \sqrt{A_1\, A_2}\right].$$

27. Sphere. $\text{Vol.} = \tfrac{4}{3}\pi r^3$.

Zone of Sphere. $\text{Vol.} = \tfrac{1}{6}\pi h\left\{h^2 + 3\,(r_1{}^2 + r_2{}^2)\right\}$, where $r_1\, r_2$ are the radii of the ∥ ends and h is the $\perp^{\text{r}}$ dist. between them.

Note.—If $x_1\, x_2$ be distances of the parallel ends from the centre and a be the radius of the sphere this result may also be written

$$\tfrac{1}{3}\pi\,(x_1 - x_2)\left\{3a^2 - (x_1{}^2 + x_1 x_2 + x_2{}^2)\right\}.$$

Segment of Sphere. Put $r_2 = 0$ in 'zone'.

28. Ellipsoid. $\text{Vol.} = \tfrac{4}{3}\pi abc$.

Spheroid. Put $b = c$.

29. Paraboloid of Revolution. The volume of a paraboloid of revolution cut off by a plane $\perp^{\text{r}}$ to the axis = half that of the cylinder on the same plane base and of the same height.

30. Similar Solids. Volumes proportional to the cubes of corresponding edges.

31. Pappus's (Guldinus's) Theorems.

(1) If an arc of a plane curve be made to revolve round any line in its plane which does not intersect it, the area of the surface thus formed = length of arc × length of path described by 'centroid' of arc.

(2) If a closed plane curve be made to revolve round any line in its plane which does not intersect it, the volume of the solid thus formed = area of curve × length of path described by the 'centroid' of area.

Note.—Hence surface and volume of *Anchor Ring.*

ALGEBRA

1. Multiplication and Division.

(1) Multiplication and Division should usually be performed using coefficients only.

Ex. $3x^2-x+1$ is written $3-1+1$, $-a^2+2ab+3b^2$ as $-1+2+3$.

(2) *Horner's Synthetic Division.*

Ex. 1. Divide x^5+1 by x^2-2x+1

	$1+0+0+0$	$+0+1$
2	$2+\mathbf{4}+6$	$+\mathbf{8}$
-1	$-1-\mathbf{2}$	$-3-\mathbf{4}$
	$1+2+3+4$	$+5-3$

The divisor is written to L. of L.H. vertical line without its first term (if the coefficient of this is 1) and with the *sign* of the rest *changed*. Subtraction replaced by addition.

Ans. x^3+2x^2+3x+4, and remainder $5x-3$.

Ex. 2. Divide $2x^4+x^3y-4x^2y^2+5xy^3+y^4$ by $2x^2-xy+y^2$ as far as the term involving $1/x^2$.

	$2+1-4+5+1$	
1	$1+\mathbf{1}-2+\mathbf{1}$	
-1	$-1-\mathbf{1}+2$	
2	$2+2-4+2+4$	
	$1+1-2+1+2$	

First term of divisor placed as divisor in last line but one and used before any term of that line is used as a multiplier. Other terms of divisor have signs changed as usual. In practice the last line but one can be omitted.

Ans. $x^2+xy-2y^2+\frac{y^3}{x}+\frac{2y^4}{x^2}+\ldots$

[See also *TH. EQNS.* § 1.]

(3) $\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots$ exactly, always, n integral.

$\frac{x^n+a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots$ but *always with remainder* $2a^n$.

$$\frac{x^n - a^n}{x + a} = x^{n-1} - ax^{n-2} + a^2x^{n-3} - \dots \quad \text{exactly, } n \text{ even,}$$

remainder $-2a^n$, n odd.

$$\frac{x^n + a^n}{x + a} = x^{n-1} - ax^{n-2} + a^2x^{n-3} - \dots \quad \text{exactly, } n \text{ odd,}$$

remainder $2a^n$, n even.

Proof. By Induction.

2. Identities.

(1) $(x+a)(x+b) = x^2 + (a+b)x + ab,$
$(x+a)(x+b)\dots(n \text{ factors})$
$= x^n + x^{n-1}\Sigma a + x^{n-2}\Sigma ab + \dots + abc\dots.$

(2) $(a+b)(a-b) = a^2 - b^2$, i.e. sum × diff. = diff. of squares,
$(a \pm b)^2 = a^2 \pm 2ab + b^2,$
$(\Sigma a)^2 = \Sigma a^2 + 2\Sigma ab$
= sum of squares + twice product of each term into sum of all that follow it.

(3) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$
$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b)$
$= \Sigma a^3 + 3\Sigma a^2b + 6abc,$
$(\Sigma a)^3 = \Sigma a^3 + 3\Sigma a^2b + 6\Sigma abc.$

(4) $(a+b)^n = a^n + na^{n-1}b + \dfrac{n(n-1)}{2!}a^{n-2}b^2 + \dots$ [See § 19.]
$(a-b)^n = a^n - na^{n-1}b + \dfrac{n(n-1)}{2!}a^{n-2}b^2 - \dots.$

The coefficients are determined by **Newton's Rule**, which is shown in the diagram. Each number in the diagram is the sum of the number immediately above it and of the number immediately to the left of that. The diagram gives the cofficients up to $(a+b)^5$.

1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	

(5) $a^3+b^3=(a+b)(a^2-ab+b^2)=(a+b)(a+\omega b)(a+\omega^2 b)$,
$a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)(a-\omega b)(a-\omega^2 b)$,
$a^3+b^3+c^3-3abc=(a+b+c)(\Sigma a^2-\Sigma bc)$
$=(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$,
$\Sigma a^3-3\Sigma abc=(\Sigma a)(\Sigma a^2-\Sigma ab)$.

Note.—For ω, ω^2 see § 12 (8).

(6) $a^4+a^2b^2+b^4=(a^2-ab+b^2)(a^2+ab+b^2)$,
$2\Sigma b^2c^2-\Sigma a^4=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

(7) $\Sigma(b-c)=0$, $\Sigma a(b-c)=0$,
$\Sigma bc(b-c)$, $\Sigma a^2(b-c)$, $\Sigma a(b^2-c^2)$
all $=\pm(b-c)(c-a)(a-b)$,
$\Sigma bc(b^2-c^2)$, $\Sigma a^3(b-c)$, $\Sigma a(b^3-c^3)$
all $=\pm(b-c)(c-a)(a-b)(a+b+c)$.

The sign of the ambiguity is determined by picking out a particular term on both sides.

(8) $(b+c)(c+a)(a+b)=\Sigma a^2b+2abc$,
$(a+b+c)(bc+ca+ab)=\Sigma a^2b+3abc$.

(9) $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\Sigma(b-c)^2$,
$(a^2+b^2)(x^2+y^2)=(ax+by)^2+(ay-bx)^2$,
$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$
$=(bz-cy)^2+(cx-az)^2+(ay-bx)^2$.

3. Factorization.

(1) *The Remainder Theorem.* If $f(x)$, a rational, integral, algebraical function of x, is divided by $x-\alpha$ the remainder is $f(\alpha)$. [*TH. EQNS.* § 1.] Hence $f(x)$ is divisible by $x-\alpha$ if $f(\alpha)=0$.

(2) To factorize an expression the following methods are available:

(*a*) Remove any obvious factors.

(*b*) If the expression contains any letter a in the first power only, arrange it as $Ma+N$ and examine H. C. F. of M and N.

(*c*) If no letter occurs in the first power only, arrange in terms of a letter occurring in the second, as Ma^2+Na+P,

and examine if M, N, P have a common factor, or solve the quadratic $Ma^2 + Na + P$ for a, obtaining factors

$$M(a-\alpha)(a-\beta). \qquad [\S\ 8\ (3).]$$

(*d*) Obtain some relation which makes the expression vanish. Thus if it vanish when $b = c$, $b - c$ is a factor.

(*e*) Having obtained one factor, examine the symmetry of the expression so as to deduce others, e. g. from $b - c$ also often $c - a$, $a - b$.

(3) To factorize $ax^2 + bx + c$ when $a \neq 1$. [See § 8 (3).] Split ac into two factors, p, q, whose sum is b. Write the expression $ax^2 + px + qx + c$. The simplest algebraical factor of $qx + c$ gives one factor. The other is written down immediately.

(4) If more factors are obtainable than the degree of the expression warrants it vanishes identically.

(5) $\Sigma a^4(b-c)$, $\Sigma a^3(b^2-c^2)$, $\Sigma a^2(b^3-c^3)$, $\Sigma a(b^4-c^4)$, $\Sigma bc(b^3-c^3)$ are all of the form

$$(b-c)(c-a)(a-b)[p\Sigma a^2 + q\Sigma bc].$$

Determine p, q by putting $a = 0$.

4. Identities.

(1) If $F(a, b, c) = a^3 + b^3 + c^3 - 3abc$,

$$F(a, b, c) \cdot F(x, y, z) = F(A, B, C),$$

where $A = ax + cy + bz$, &c.; [Use § 2 (5)]

$$\{F(a, b, c)\}^2 = F(a^2 + 2bc,\ b^2 + 2ca,\ c^2 + 2ab)$$
$$= F(a^2 - bc,\ b^2 - ca,\ c^2 - ab).$$

(2) Identities involving $a + b + c = 0$ may be dealt with in four ways:

(*a*) Put $a = \beta - \gamma$, $b = \gamma - \alpha$, $c = \alpha - \beta$.

(*b*) Put $a = m + n$, $b = m\omega + n\omega^2$, $c = m\omega^2 + n\omega$.

(*c*) $(1 - ax)(1 - bx)(1 - cx) = 1 - qx^2 - rx^3$,

where $q = -(bc + ca + ab)$, $r = abc$.

Take logs. and equate coefficients of x^n to obtain Σa^n in terms of q and r.

(*d*) By *Partial Fractions*.

Thus $$\frac{1}{(1-ax)(1-bx)(1-cx)} = \sum \frac{a^2}{(a-b)(a-c)} \cdot \frac{1}{1-ax}$$

by expansion and equating coefficients of x^{n-2} gives $\Sigma a^n (b-c)$.

(3) Trigonometrical substitutions are sometimes useful. For expressions involving

$$\frac{2x}{1-x^2}, \frac{2y}{1-y^2}, \frac{2z}{1-z^2} \text{ or } \frac{2x}{1+x^2}, \text{ &c., or } \frac{1-x^2}{1+x^2}, \text{ &c.,}$$

substitute $x = \tan A$, $y = \tan B$, $z = \tan C$.

If $x+y+z = xyz$, then $A+B+C = n\pi$.

5. Vanishing Fractions.

(1) If a is a finite quantity,

$$a \times 0 = 0, \quad \frac{0}{a} = 0, \quad \frac{a}{0} = \pm\infty .$$

(2) If a fraction P/Q becomes $0/0$ for any value of x, say $x = a$, its value can sometimes be found by dividing both numerator and denominator by $x-a$.

(3) In other cases put $x = a+h$, expand in powers of h, cancel h once or more from both numerator and denominator, and put $h = 0$.

6. Indices.

(1) *Fundamental Laws.*

$$a^m \times a^n = a^{m+n}, \quad (a^m)^n = a^{mn}, \quad (ab)^m = a^m b^m.$$

(2) *Derived Laws.* $$\frac{a^m}{a^n} = a^{m-n}, \quad a^0 = 1 \text{ (if } a \text{ is finite),}$$

$$a^{\frac{1}{m}} = \sqrt[m]{a}, \quad a^{-n} = \frac{1}{a^n}.$$

7. Simple Simultaneous Equations. [See also *DETERM.* § 13.] The rule of **Cross-Multiplication** enables us to write down the answer. Thus for

$$\begin{cases} ax+by+cz = 0, \\ px+qy+rz = 0, \end{cases}$$

we have $$\frac{x}{br-cq} = \frac{y}{cp-ar} = \frac{z}{aq-bp}.$$

The denominators are written by the diagonals of the following $\begin{vmatrix} b, & c \\ q, & r \end{vmatrix}$, $\begin{vmatrix} c, & a \\ r, & p \end{vmatrix}$, $\begin{vmatrix} a, & b \\ p, & q \end{vmatrix}$.

8. Quadratic Equations. $ax^2 + bx + c = 0$.

(1) Solve where possible by factorization. If one root is known to be α, the other is c/α.

(2) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Roots *real*, *equal*, or *complex*, as $b^2 >$, $=$, or $< 4ac$.

(3) $\alpha + \beta = -\dfrac{b}{a}$, $\quad \alpha\beta = \dfrac{c}{a}$.

The equation is $a(x - \alpha)(x - \beta) = 0$.

Ex. To find the equation whose roots are $1 + \beta/\alpha$, $1 + \alpha/\beta$.
$y = 1 + \beta/\alpha = (\alpha + \beta)/\alpha = -b/a\alpha$, $\therefore \alpha = -b/ay$.
But $a\alpha^2 + b\alpha + c = 0$ $\therefore$ &c.

Caution. When involution has occurred in solving any equation the roots obtained may not satisfy the equation and *must* be tested.

(4) The expression $ax^2 + bx + c$ has always the same sign as a unless α and β are real and different and x lies between them.

(5) *Maxima and Minima* of $ax^2 + bx + c$. Write it

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Or put $ax^2 + bx + c = y$, solve for x and express condition that roots are real.

Maxima and Minima of $(ax^2 + bx + c)/(px^2 + qx + r)$. Put exp. $= y$, solve for x and express condition that x is real. Then use (4) above.

(6) *Eliminant of two Quadratics.* To eliminate x between $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, use § 7 above to obtain

$$\frac{x^2}{br - cq} = \frac{x}{cp - ar} = \frac{1}{aq - bp}.$$

9. Equations, Numerical Solution. [See also *TH. EQNS.* § 8.] Graphical methods are available if approximate answers only are required, and can be applied to:

(*a*) Simple Simultaneous Equations in two variables. The point of intersection of the straight lines $ax+by+c=0$, $px+qy+r=0$, will give the solution of these two equations taken simultaneously.

(*b*) Quadratics. For $ax^2+bx+c=0$ find the intersections of $y=x^2$ and $ay+bx+c=0$.

(*c*) Cubics and all higher equations. Thus $x^3+bx+c=0$ can be solved by $y=x^3$, $y+bx+c=0$.

Note.—The curves $y=x^2$, $y=x^3$ can be obtained accurately cut in celluloid. (Messrs. Macmillan, or J. J. Griffin.)

10. Equations, Artifices in Solution.

(1) $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}=\dfrac{x+c}{x-c}+\dfrac{x+d}{x-d}$. Divide in each fraction: $\dfrac{x+a}{x-a}=1+\dfrac{2a}{x-a}$.

(2) $ax^2+\sqrt{ax^2-bx+c}=bx+d$. Put $ax^2-bx+c=y^2$.

(3) $\dfrac{m}{x}+\dfrac{n}{y}=a$, $\dfrac{p}{x}+\dfrac{q}{y}=b$. First solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$.

(4) $\left.\begin{matrix}x+y=a\\xy=b\end{matrix}\right\}$; $\therefore$ x, y are roots of $\xi^2-a\xi+b=0$.

(5) $\left.\begin{matrix}x^2\pm y^2=a\\x\pm y=b\end{matrix}\right\}$, $\left.\begin{matrix}xy=a\\x\pm y=b\end{matrix}\right\}$, $\left.\begin{matrix}x^2+y^2=a\\xy=b\end{matrix}\right\}$.

Determine $x+y$ and $x-y$.

$\left.\begin{matrix}x^2-y^2=a\\xy=b\end{matrix}\right\}$. Use (6).

(6) If 'homogeneous' in x and y put $y=mx$ and find m.

(7) $\left.\begin{matrix}x^3+y^3=a\\x+y=b\end{matrix}\right\}$. Use $(x+y)^3=x^3+y^3+3xy(x+y)$ to find xy, then (4). Similarly for $\left.\begin{matrix}x^3-y^3=a\\x-y=b\end{matrix}\right\}$.

(8) $yz = a^2,\ zx = b^2,\ xy = c^2\ ;\ \therefore\ xyz = \pm abc.$

(9) $x(x+y+z) = a$ and two similar equations ;

$$\therefore\ x+y+z = \pm\sqrt{a+b+c}.$$

(10) $\left.\begin{aligned} x^2-yz &= a \\ y^2-zx &= b \\ z^2-xy &= c \end{aligned}\right\}$, $bc-a^2 = x(3xyz-x^3-y^3-z^3)$;

$$\therefore\ \frac{x}{bc-a^2} = \frac{y}{ca-b^2} = \frac{z}{ab-c^2} = \pm\sqrt{\frac{yz-x^2}{a(3abc-a^3-b^3-c^3)}} = \&c.$$

(11) $\left.\begin{aligned} x^2+2yz &= a \\ y^2+2zx &= b \\ z^2+2xy &= c \end{aligned}\right\}$; $\quad\therefore\quad x+y+z = \pm\sqrt{a+b+c},$

$$x+\omega y+\omega^2 z = \pm\sqrt{a+\omega^2 b+\omega c},$$

$$x+\omega^2 y+\omega z = \pm\sqrt{a+\omega b+\omega^2 c}.$$

Add these equations and remember $1+\omega+\omega^2 = 0$.

(12) $\left.\begin{aligned} x+y+z &= a \\ x^2+y^2+z^2 &= b \\ x^3+y^3+z^3 &= c \end{aligned}\right\}$. Determine $yz+zx+xy = q,\ xyz = r$. Then x, y, z are roots of $\xi^3 - a\xi^2 + q\xi - r = 0$.

11. Equations of Higher Orders. [See also § 9, and *TH. EQNS.*]

(1) If the roots of $p_0 x^n + p_1 x^{n-1} + \ldots + p_n = 0$ are $\alpha_1, \alpha_2, \ldots \alpha_n$, the equation is $p_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) = 0$, and

$$\alpha_1+\alpha_2+\ldots+\alpha_n = -p_1/p_0,$$

$$\Sigma\alpha_1\alpha_2 = p_2/p_0,\ \&c.,$$

$$\alpha_1\alpha_2\ldots\alpha_n = (-)^n p_n/p_0.$$

(2) If $\sqrt{a}+\sqrt{b}$ is a root, and coefficients are rational, $\pm\sqrt{a}\pm\sqrt{b}$ are also roots.

(3) *Reciprocal Equations.* [See also *TH. EQNS.* § 10.]

(*a*) If of odd degree test for roots 1 or -1.

(*b*) If of even degree, say $x^4+ax^3+bx^2+ax+1 = 0$, divide by x^2, and put $x+\frac{1}{x} = \xi,\ x^2+\frac{1}{x^2} = \xi^2-2$.

(*c*) Similar method for $x^4+ax^3+bx^2+ak^2x+k^4 = 0$.

(4) If no other method is available endeavour to guess

a root $x = \alpha$ (guided by the consideration that it is a factor of the last term), and divide by $x - \alpha$.

(5) If elimination is required between two higher equations, Bézout's Method is recommended. [*TH. EQNS.* § 16.]

12. Irrational and Complex Expressions.

(1) If $a + \sqrt{b} = c + \sqrt{d}$, $a = c$, $b = d$, provided these quantities are known to be rational and b and d are not squares.

(2) If $a + bi = c + di$, $a = c$, $b = d$, provided these quantities are known to be real. [$i = \sqrt{-1}$.]

(3) An identity containing i remains true if $-i$ be written for i, provided that the coefficients involved are known to be real.

(4) $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$ is satisfied if x and y are so chosen that $x + y = a$, $4xy = b$.

(5) $\sqrt[3]{a + \sqrt{b}}$. If this can be expressed as $x + \sqrt{y}$, then $x^3 + 3xy = a$. But also $\sqrt[3]{a^2 - b} = (x + \sqrt{y})(x - \sqrt{y})$. Eliminate y to determine x.

(6) $\sqrt{p + \sqrt{q} + \sqrt{r} + \sqrt{s}}$ can be expressed as

$$\sqrt{x} + \sqrt{y} + \sqrt{z}$$

if it is possible to find x, y, z to satisfy the *four* equations

$$x + y + z = p, \quad 2\sqrt{yz} = p, \quad 2\sqrt{zx} = q, \quad 2\sqrt{xy} = r.$$

(7) *Rationalizing Factor.* To rationalize $a^{\frac{1}{p}} \pm b^{\frac{1}{q}}$: let $a^{\frac{1}{p}} = x$, $b^{\frac{1}{q}} = y$, and let n be L. C. M. of p and q. Divide $x^n \pm y^n$ by $x \pm y$ and substitute.

(8) *Cube Roots of Unity.* The complex cube roots of 1 are $\frac{-1 \pm i\sqrt{3}}{2}$. They are denoted by ω, ω^2 and have the following properties:

(*a*) Either is the square of the other.

(*b*) $1 + \omega + \omega^2 = 0$. (*c*) $\omega^3 = 1$.

13. Ratio and Proportion.

(1) If $a/b = c/d$,

$$a+b : b = c+d : d,$$
$$a-b : b = c-d : d,$$
$$a+b : a-b = c+d : c-d,$$
$$pa+qb : pa-qb = pc+qd : pc-qd.$$

(2) If $a/b = c/d$, each $= \left(\frac{pa^n+qc^n}{pb^n+qd^n}\right)^{\frac{1}{n}}$.

(3) If a_1/b_1, a_2/b_2, a_3/b_3 ... a_n/b_n are in ascending order of magnitude,

$$\frac{a_n}{b_n} > \frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n} > \frac{a_1}{b_1}.$$

(4) A fraction is brought nearer to unity by adding the same (positive) quantity to both numerator and denominator.

14. Variation.

(1) If $a \propto b$, $a = mb$.

(2) If $a \propto$ inversely as b, $a \propto 1/b$, $a = m/b$.

(3) If $a \propto b$ when c is constant, and $\propto c$ when b is constant, then when both vary $a \propto bc$.

15. Scales of Notation.

(1) A number in scale r is divisible by $r-1$ if the sum of the digits $p_0+p_1+p_2+\ldots$ is so divisible.

(2) A number is divisible by $r+1$ if $p_0-p_1+p_2-\ldots$ is so divisible.

16. Progressions.

(1) *Arithmetical.* $l = a+(n-1)\,d$,

$$s = \frac{n}{2}(a+l) = \frac{n}{2}(2a+\overline{n-1}\,.\,d).$$

Mean (A) between a and b is $\frac{1}{2}(a+b)$.

(2) *Geometrical.* $l = ar^{n-1}$,

$$s = a\frac{r^n-1}{r-1} = a\cdot\frac{1-r^n}{1-r} = \frac{lr-a}{r-1},$$

$$s_\infty = \frac{a}{1-r}.$$

Mean (G) between a and b is $\sqrt{ab}$.

Note.—The proof given for s_∞ is usually slipshod. It is *not* obvious that the limit of $a^n = 0$ $(n = \infty)$ if $r < 1$. Cf. the second limit in § 22 (4) when $a = -1$. A geometrical illustration is easy and valuable.

(3) *Harmonical.* If a, b, c are in H. P., $1/a$, $1/b$, $1/c$ are in A. P. $\quad a : c = a - b : b - c.$

Mean (H) between a and b is $\dfrac{2ab}{a+b}$.

Note.—To construct H geometrically take $AM = a, MB = b$ in a straight line, and at A and B erect $\perp^{rs}$ $AK = a$, $BL = b$. Join KL. At M erect $MN \perp^r AB$ meeting KL in N. Then $MN = H$.

(4) $A > G > H$ and $G^2 = AH$.

17. Interest. If P be the principal, r = rate per cent., $R = 1 + (r/100)$.

Simple Interest for n years $= Pnr/100$.

Compound ,, ,, ,, $= PR^n - P$.

18. Permutations or Arrangements.

(1) $P_r^n = n(n-1) \ldots (n-r+1) = \dfrac{n!}{(n-r)!}$; $P_n^n = n!$

(2) Permutations of n things, p alike, q alike, &c.

$$= \frac{n!}{p!\,q!\,r! \ldots}.$$

(3) Permutations of n things, r together, repetitions allowed $= n^r$.

Combinations or Selections.

(1) $C_r^n = \dfrac{n(n-1) \ldots (n-r+1)}{r!} = \dfrac{n!}{(n-r)!\,r!} = C_{n-r}^n$.

(2) Total number of selections from n things $= 2^n - 1$.

(3) Selections of n things, r together, repetitions allowed, $= C_r^{n+r-1}$.

(4) *Vandermonde's Theorem.* [*Proof* § 19 (4) (*a*).]

$$C_r^{m+n} = C_r^m + C_{r-1}^m . C_1^n + C_{r-2}^m . C_2^n + \ldots + C_1^m C_{r-1}^n + C_r^n.$$

(5) No. of selections, r together, from objects p alike of one kind, q of another, &c., is the coefficient of x^r in .

$$\frac{1-x^{p+1}}{1-x} \cdot \frac{1-x^{q+1}}{1-x} \cdot \dots .$$

19. Binomial Theorem.

(1) $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \dots,$

$(1-x)^{-1} = 1 + x + x^2 + \dots$ to ∞, [C. if $x < 1$

$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$ to ∞. [C. if $x < 1$

Proof.—When n is not integral for $(1+x)^n$. Prove Vandermonde's Theorem (§ 17 (4)) when m, n, r are integral. Giving to C_r^m the meaning $m(m-1)\dots(m-r+1)/r!$ point out that each side of the Vandermonde equation is of r-degree in n, that the two sides are equal if m have any integral value for *all* integral values of n. ∴ they are equal altogether for any value of n (integral or otherwise). So for m. Let $f(m) = 1 + mx + m(m-1)x^2/2! + \dots$. Show that f converges if $x < 1$. Using Vandermonde prove $f(m) \times f(n) = f(m+n)$, &c.

(2) $(r+1)$th term of $(1+x)^n$

$$= \frac{n(n-1)\dots(n-r+1)}{r!}x^r = C_r^n x^r,$$

$(r+1)$th term of $(1+x)^{-n}$

$$= \frac{n(n+1)\dots(n+r-1)}{r!}(-x)^r = (-)^r H_r^n . x^r.$$

Thus in $(1-x)^{-n}$ all the terms are $+^{ve}$.

Number of terms $= n+1$ if n is a positive integer.

Middle term, n even, $r = \frac{n}{2}$; n odd, $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$.

(3) *Homogeneous Products* of rth degree of n quantities,

$$H_r^n = \frac{n(n+1)\dots(n+r-1)}{1.2\dots r}.$$

Proof.—Consider H. P. of degree 7 of 4 letters a, b, c, d. Denote a^3bd^3 by $\dots|.||\dots$ The vertical strokes divide the 7 dots into groups, first group for a's &c., none in the c-group. H_4^7 is the no. of ways of arranging the 10 symbols, 7 dots, 3 strokes. The reasoning is perfectly general. (Whipple.)

The sum of these products is the coefficient of x^r in

$$1/(1-ax)(1-bx)(1-cx)\ldots.$$ [See § 4 (2) (*d*)

(4) *Derived Binomial Series* are obtained by equating coefficients in such identities as

(*a*) $(1+x)^m \times (1+x)^n = (1+x)^{m+n}$, [giving Vandermonde's Theorem.

$$(1+x)^m \times (1-x)^m = (1-x^2)^m,$$

$$(1+x)^m \times (1-x)^{-n} = \frac{(1+x)^m}{(1-x)^n} = \frac{(1-x+2x)^m}{(1-x)^n}, \text{ \&c.}$$

(*b*) $\{1-(5x-6x^2)\}^{-1} = 3(1-3x)^{-1} - 2(1-2x)^{-1}$.

20. Multinomial Theorem. The general term of

$$(a_0 + a_1 x + a_2 x^2 + \ldots)^n$$

is $$\frac{n(n-1)\ldots(n-k+1)}{q!\,r!\,s!\ldots} a_0^{n-k} a_1^q a_2^r a_3^s \ldots x^{q+2r+3s+\cdots},$$

where $\mu = q+r+s+\ldots$.

Note.—Wherever possible avoid the use of this theorem. Thus $(1+2x+3x^2+\ldots)^n = (1-x)^{-2n}$.

21. Logarithms.

(1) $\log xy = \log x + \log y$, $\quad \log \frac{x}{y} = \log x - \log y$,

$\log x^r = r \log x$, $\quad \log \sqrt[r]{x} = \frac{1}{r} \log x$.

(2) $\log_a b \times \log_b a = 1$,

$\log_a x = \log_a b \times \log_b x$.

Mnemonic.— $x/a = b/a \times x/b$.

(3) If the base is 10 the characteristic of a number with n figures in its integral part is $n-1$; of a number with n ciphers before a significant figure is reached after the decimal point the characteristic is $-n-1$.

22. Exponential and Logarithmic Series.

(1) *Exponential Theorem.*

$$e = 1+1+1/2!+1/3!+\ldots = 2{\cdot}71\ldots,$$

$$e^x = 1+x+x^2/2!+x^3/3!+\ldots,$$

$$a^x = 1+x\log_e a+(x\log_e a)^2/2!+\ldots.$$

Proof.—Let $f(x) = 1+x+x^2/2!+x^3/3!+\ldots$. Prove the series always convergent. Using Binomial Theorem deduce $f(m)\times f(n) = f(m+n)$. Deduce $f(m) = [f(1)]^m = e^m$, m integral. Prove next for $f(p/q)$. Last prove for $f(-m)$.

(2) *Logarithmic Series.*

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, \quad [\text{C. if } x<1$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots. \quad [\text{C. if } x<1$$

Proof.—Equate coefficients of y in $(1+x)^y = e^{y\log(1+x)}$. Equating coefficients of y^2 will give a series for $\{\log(1+x)\}^2$.

(3) *Series for Calculation.* From (2) obtain $\log\dfrac{1+x}{1-x}$. Put $x = 1/(2n+1)$, and hence

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \ldots\right\}.$$

Note.—To calculate $\log_{10} 2$. Put $n = 1$ and find $\log_e 2$; thence $\log_e 3$, then $\log_e 9$, thence $\log_e 10$, $\log_{10} e$, $\log_{10} 2$.

(4) *Limits.* $\mathrm{Lt}_{n=\infty}\left(1+\dfrac{1}{n}\right)^n = e$, $\mathrm{Lt}_{n=\infty}\left(1+\dfrac{a}{n}\right)^n = e^a$,

$$\mathrm{Lt}_{x=0}(a^x-1)/x = \log_e a.$$

(5) *Derived Logarithmic Series* are obtained by equating coefficients in such identities as:

(*a*) $\log(1-5x+6x^2) = \log(1-3x)+\log(1-2x)$.

(*b*) $(e^x-1)^n = (x+x^2/2!+\ldots)^n$

$$= e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{2!}e^{(n-2)x} - \ldots;$$

$$\therefore\quad n^n - n(n-1)^n + \frac{n(n-1)}{2!}(n-2)^n - \ldots = n!$$

(*c*) $(e^x-e^{-x})^n$ treated similarly gives

$$n^n - n(n-2)^n + \frac{n(n-1)}{2!}(n-4)^n - \ldots = 2^n \cdot n!$$

23. i. Continued Fractions.

C. F. $\dfrac{1}{a_1+}\ \dfrac{1}{a_2+}\ \dfrac{1}{a_3+}\ \ldots$ | C. F. is $\dfrac{b_1}{a_1+}\ \dfrac{b_2}{a_1+}\ \ldots$.

Convergents are $p_1/q_1,\ p_2/q_2, \ldots$.

(1) $p_n = a_n p_{n-1} + p_{n-2}$,
$q_n = a_n q_{n-1} + q_{n-2}$. | (1) $p_n = a_n p_{n-1} + b_n p_{n-2}$
$q_n = a_n q_{n-1} + b_n q_{n-2}$

Hence p_n/p_{n-1} can be expressed as C. F.

(2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ | (2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 \ldots b_n$.

This is useful as a verification test when calculating convergents. It will usually suffice to check by the units figures only.

(3) Convergents formed as in (1) are necessarily in their lowest terms.

(3) Convergents formed as in (1) will often not be in their lowest terms. They *must not* be reduced to lowest terms before continuing with (1).

(4) Every convergent is nearer to the C. F. than any preceding convergent, but they are alternately less and greater than the C. F.

(4) Since this C. F. does not necessarily tend to any definite limit, the remaining §§ are given as applying only to the first form.

(5) The error in taking p_n/q_n for the C. F. $< 1/q_n^2$ but $> 1/(2q_{n+1}^2)$.

(6) Any convergent is a nearer approximation to value of C. F. than any fraction whose denominator is not greater than that of the convergent.

(7) *Intermediate Convergents.* The series of fractions

$$(p_{n-2} + r p_{n-1})/(q_{n-2} + r q_{n-1})$$

when r has all integral values from 0 to a_n are called the Intermediate Convergents between p_{n-2}/q_{n-2} and p_n/q_n, which form the first ($r = 0$) and last ($r = a_n$) members of the series.

If P_r/Q_r, P_{r-1}/Q_{r-1} be any consecutive pair, $P_r Q_{r-1} - P_{r-1} Q_r = \pm 1$. The series of fractions $p_1/q_1 \ldots p_3/q_3 \ldots p_5/q_5 \ldots$, the gaps being filled

by the intermediate convergents, is a series continually increasing towards C. F., while the series $p_2/q_2 \ldots p_4/q_4 \ldots p_6/q_6 \ldots$ is a series continually decreasing towards the C. F. It is impossible between any consecutive pair of either series to insert a fraction of smaller denominator.

Note.—Hence the solution of such problems as : Find the fraction of denominator not more than 2 figures which is the best approximation to π. [*Ans.* 311/99.]

ii. *Recurring Continued Fractions.*

Caution.—These results apply only to $\sqrt{N}/M$ and not to $(P+\sqrt{N})/M$ unless $P = 0$.

(1) To evaluate $x = \dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{a+}\,\dfrac{1}{b+}\ldots$, write

$$x = \frac{1}{a+}\,\frac{1}{b+x}$$

and solve the quadratic.

(2) If $\sqrt{N}/M$ is converted into a C. F. it necessarily takes the form

$$a_1 + \left[\frac{1}{b_1+}\,\frac{1}{b_2+}\,\frac{1}{b_3+}\ldots+\frac{1}{b_3+}\,\frac{1}{b_2+}\,\frac{1}{b_1+}\,\frac{1}{2a_1+}\right]\frac{1}{b_1+}\ldots,$$

i. e. after a while the partial quotients reappear, but in the reverse order ; a partial quotient double the integral part will then follow ; after the appearance of $2a$, recurrence takes place and the recurring part is shown by brackets. If $a_1 = 0$ the last partial quotient of recurring period is $2b_1$. The middle quotient may or may not be repeated.

(3) If $\sqrt{N}$ is converted into a C. F. and if p_n/q_n is the penultimate convergent of any recurring period, i.e. the convergent formed from the partial quotient immediately preceding $2a_1$,

$$p_n^2 - Nq_n^2 = (-1)^n.$$

(4) p_n/q_n being the penultimate convergent of a recurring period,

$$\frac{p_{2n}}{q_{2n}} = \frac{1}{2}\left[\frac{p_n}{q_n} + \frac{Nq_n}{p_n}\right].$$

Note.—An exceptionally good method for finding square roots rapidly. Thus for $\sqrt{17}$, $p_1/q_1 = 4$, $p_2/q_2 = \frac{33}{8}$, $p_4/q_4 = \frac{2177}{528}$, the error in which does not affect the first 5 places. [See § 23, i. (5).]

iii. *Evaluation of* p_n/q_n.

(1) C. F. $\dfrac{b}{a+}\,\dfrac{b}{a+}\,\ldots$. $p_n = ap_{n-1} + bp_{n-2}$. Hence p_n is nth term of a recurring series whose scale is $1 - ax - bx^2$. [See § 30 (8) and § 31.]

(2) C. F. $\dfrac{b}{a+}\,\dfrac{d}{c+}\,\dfrac{b}{a+}\,\dfrac{d}{c+}\,\ldots$. Obtain relation connecting p_{2n}, p_{2n-2}, p_{2n-4}.

(3) C. F. $\dfrac{b_1}{a_1+}\,\dfrac{b_2}{a_2+}\,\ldots$. A general form for p_n/q_n can be found in certain cases.

Exx. (1) $\dfrac{1^2}{1+}\,\dfrac{2^2}{1+}\,\dfrac{3^2}{1+}\,\ldots$:

$$p_n = p_{n-1} + n^2 p_{n-2},\ p_n - (n+1)\,p_{n-1} = -n\,[p_{n-1} - np_{n-2}].$$

(2) $\dfrac{r^2}{1+}\,\dfrac{r(r+1)}{2+}\,\dfrac{r(r+2)}{3+}\,\ldots$:

$$p_n - (r+n)\,p_{n-1} = -r\,[p_{n-1} - \overline{r+n-1}\,p_{n-2}].$$

Note.—From any C. F. capable of evaluation thus another may be derived. Thus $\dfrac{1}{1+}\,\dfrac{2}{2+}\,\dfrac{3}{3+}\,\ldots$ gives $p_n = np_{n-1} + np_{n-2}$. Put $p_n = P_n/n!$ We get $P_n = n^2 P_{n-1} + n^2(n-1)\,P_{n-2}$; $\therefore$ the fraction $\dfrac{2^2.1}{2^2+}\,\dfrac{3^2.2}{3^2+}\,\dfrac{4^2.3}{4^2+}\,\ldots$ can be evaluated. So also we may put $p_n = P_n/r^n$ where r is any integer.

iv. *Conversion of Series into C. F.*

(1) $$\frac{1}{u_1} + \frac{1}{u_2} + \ldots + \frac{1}{u_n} = \frac{1}{u_1}\,\frac{u_1^2}{u_1+u_2-}\,\frac{u_2^2}{u_2+u_3-}\,\ldots\,-\frac{u_{n-1}^2}{u_{n-1}+u_n}.$$

Proof.—To recall this solve the equation $1/u_r + 1/u_{r+1} = 1/(u_r + x_r)$ giving $x_r = -u_r^2/(u_r + u_{r+1})$.

(2) The following are added for reference (deduced from (1)) :

$$\frac{1}{u_1}-\frac{1}{u_2}+\ldots = \frac{1}{u_1+}\ \frac{u_1^2}{u_2-u_1+}\ \frac{u_2^2}{u_3-u_2+}\ldots$$

$$\frac{1}{u_0}+\frac{x}{u_1}+\frac{x^2}{u_2}+\ldots = \frac{1}{u_0-}\ \frac{u_0^2}{u_0x+u_1-}\ \frac{u_1^2}{u_1x+u_2-}\ldots$$

$$\frac{a_1}{b_1}x+\frac{a_1a_2}{b_1b_2}x^2+\ldots = \frac{a_1x}{b_1-}\ \frac{b_1a_2x}{b_2+a_2x-}\ \frac{b_2a_3x}{b_3+a_3x-}\ldots$$

(3) The quotient of two convergent series S_1, S_2 may be expressed as follows :

$$S_1 = 1+p_1x+p_2x^2+\ldots \qquad S_2 = 1+q_1x+q_2x^2+\ldots$$

$S_2-S_1 = (q_1-p_1)xS_3$, where $S_3 = 1+r_1x+r_2x^2+\ldots$;

$$\therefore\ \frac{S_1}{S_2} = 1-(q_1-p_1)x\frac{S_3}{S_2}.$$

$S_3-S_2 = (r_1-q_1)xS_4$, where $S_4 = 1+t_1x+t_2x^2+\ldots$;

$$\therefore\ \frac{S_2}{S_3} = 1-(r_1-q_1)x\frac{S_4}{S_3};$$

$$\therefore\ \frac{S_1}{S_2} = 1-\frac{(q_1-p_1)x}{1-}\ \frac{(r_1-q_1)x}{1-}\ \ldots.$$

If $S_2 = 1$ this gives a second form for S_1.

The method is applicable to **Gauss's Hypergeometric Series,**

$$F(\alpha,\beta,\gamma,x) = 1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots,$$

and gives the expansion of $F(\alpha,\beta+1,\gamma+1,x)/F(\alpha,\beta,\gamma,x)$. A valuable result is

$$\tan x = \frac{x}{1-}\ \frac{x^2}{3-}\ \frac{x^2}{5-}\ldots,$$

from which it can be deduced that π, π^2 are incommensurable.

v. *Convergency of C. F.* $\dfrac{b_1}{a_1+}\ \dfrac{b_2}{a_2+}\ldots$.

If the C. F. is reduced to the form $\dfrac{1}{d_1+}\ \dfrac{1}{d_2+}\ldots$ and if all the d's be + the C. F. can never be D, but will oscillate if $d_1+d_3+\ldots$ and $d_2+d_4+\ldots$ be both C, and will be C if one or both of these series be D. [Chrystal, c. xxxiv.]

Ex. $\dfrac{2}{1+}\ \dfrac{2^2}{1+}\ \dfrac{2^3}{1+}\ldots$ oscillates, the odd convergents decreasing towards 1 . 13 ..., the even convergents increasing towards · 63 ....

24. Indeterminate (Diophantine) Equations.

(1) $ax-by = c$. If α, β be one solution, the general solutions are $x = \alpha+bt$, $y = \beta-at$. The number of positive

integral solutions is infinite, unless a and b both contain a factor which does not occur in c, in which case there are no solutions.

To find α, β:

Method i. *Ex.* $7x-17y=27$; $\therefore\ x-2y-\frac{3y}{7}=4-\frac{1}{7}$; $\therefore\ (3y-1)/7=\text{integer}=z$; $\therefore\ 3y-7z=1$. Repeat process if necessary. [*Ans.* 16, 5.]

Method ii. Convert $\frac{7}{17}$ into C.F. *Penultimate convergent gives solution.* Here it is $\frac{2}{5}$. 5, 2 are solutions to $7x-17y=1$; $\therefore$ 135 $(=27\times5)$, 54 are solutions to given equation.

(2) $ax+by=c$. If α, β is one solution, the general solutions are $x=\alpha+bt$, $y=\beta-at$. The number of positive integral solutions is nearest integer to c/ab (±1, in certain cases), unless as in (1).

To find α, β proceed as in (1). Both methods are applicable. The sign of one of the terms in the convergent must be changed.

(3) $ax+by+cz=d$, $a'x+b'y+c'z=d'$ } Eliminate z and use (1) or (2). Solutions found for x, y may not all give integral solutions for z. If α, β, γ be any solution, the general solution is

$$x=\alpha+(bc'-b'c)t, \text{ \&c.}$$

(4) $ax+by+cz=d$. To find a solution use (1) Meth. i. The general solution is $x=\alpha+(b-c)p+a(b-c)q$, where p, q are any integers. *Or* give a series of values 0, 1, 2, ... to z.

(5) (*a*) $x^2-a^2y^2=\pm H$ can sometimes be solved by $x-ay=p$, $x+ay=q$, where $pq=\pm H$.

(*b*) $x^2+a^2y^2=H$, and generally $x^2+Cy^2=H$, can only be solved by trial, if at all, unless H is a square. [See (*g*) below.]

(*c*) $x^2-Cy^2=1$ [**Pellian equation**] has always an infinite number of solutions unless C is a square. [See § 23, ii (3).]

(*d*) $x^2-Cy^2=-1$ has an infinite number of solutions

provided there is an odd number of quotients in the recurring period of $\sqrt{C}$. [See § 23, ii (3).]

Note.—If one solution (p, q) to either (c) or (d) is known, an infinite number of others may be written down for $x^2 - Cy^2 = (p^2 - Cq^2)^n$ [n even for (c), odd for (d)]. ∴ solutions are found from

$$x - \sqrt{C}.y = (p - \sqrt{C}.q)^n, \quad x + \sqrt{C}.y = (p + \sqrt{C}.q)^n.$$

(e) $x^2 - Cy^2 = \pm H$ can be solved in special cases and is insoluble in others. When one solution (p, q) can be found an infinite number can also, for if (r, s) be any solution of $x^2 - Cy^2 = 1$, $(p^2 - Cq^2)(r^2 - Cs^2) = H$;

$$\therefore (pr \pm Cqs)^2 - C(ps \pm qr)^2 = H;$$

∴ a general solution is $x = pr \pm Cqs$, $y = ps \pm qr$.

Note.—The convergents to $\sqrt{C}$ will often give a series of solutions. Thus with $x^2 - 13y^2 = H$ the various convergents give an infinite series of solutions when $H = \pm 1, \pm 3, \pm 4$. Solutions exist only if $H^2 \equiv 1, 3$ or $9 \pmod{13}$, and then always and only if k can be found $(< H)$ such that $kH + 13$ is a square, and that $x^2 - 13y^2 = k$ can be solved. (Chrystal, c. xxxiii, 18.)

(f) *Ex.* i. $x^2 + 2xy - 17y^2 + 72y - 73 = 0$;

∴ $(x + y)^2 - 2(3y - 6)^2 = 1$. Solve $X^2 - 2Y^2 = 1$, and select values of Y multiples of 3. *Or* solve original equation in x. Express that quantity under radical is square. See Ex. iii.

Ex. ii. $xy - 17y^2 + 2x + 72y - 88 = 0$;

∴ $x = 17y - 106 + 300/(y + 2)$. Hence $y + 2$ is one of factors of 300. Solutions: (3, 2), (5, 3), (11, 1), (12, 4), &c.

Ex. iii. Find values of x making

$$3x^2 + 8x + 1 \text{ square} = z^2;$$

∴ $(3x + 4)^2 - 13 = 3z^2$, i.e. $y^2 - 3z^2 = 13$. One solution, $y = 4$, $z = 1$. Hence $x = 0, 4, 12, 72, 184, \ldots$.

(g) $x^2 + y^2 = z^2$ (**Pythagorean Equation**). All possible solutions are included in

$$x = \lambda(m^2 - n^2), \quad y = 2\lambda mn, \quad z = \lambda(m^2 + n^2),$$

where λ, m, n are any integers.

25. Partial Fractions. See *DIFF. CALC.* § 4.

26. Inequalities.

(1) $(x-y)^2$ is + ; $\therefore\ x^2+y^2>2xy$.

(2) See § 13 (3). $a_n/b_n > \Sigma a/\Sigma b > a_1/b_1$.

(3) A.M. of any number of + quantities > G.M.

Proof.—$(x+y)^2 = (x-y)^2+4xy$. Hence if any two of the quantities are unequal we increase their product without altering sum by making them equal.

(4) $(\Sigma a^m)/n > \{(\Sigma a)/n\}^m$, where Σa extends to the n quantities $a, b, c, \ldots$.

Proof.—First prove for 2 letters. Expand

$$(a^m+b^m)/2 = \{(a+b)/2+(a-b)/2\}^m+\ldots.$$

Three cases, (i) m + integer or −, (ii) m between 0 and 1, (iii) m not integral but +, which is deduced from case ii. Then extend to > 2 letters. If any two are unequal replace by $(a+b)/2$, then Σa unaltered, Σa^m diminished ; $\therefore$ min. val. Σa^m is when all letters are equal.

(5) If $p > q$,

(*a*) $(x^p-1)/p > (x^q-1)/q$.

(*b*) $\left(1+\frac{x}{p}\right)^p > \left(1+\frac{x}{q}\right)^q$.

27. Maxima and Minima.

(1) See § 8 (5).

(2) $(a+x)^m(b-x)^n$. Apply § 26 (3) to

$$\{(a+x)/m\}^n\ \{(b-x)/n\}^n.$$

(3) $a^m b^n c^p \ldots$, subject to the condition $a+b+c+\ldots$ = constant. Apply § 26 (3) to $(a/m)^m(b/n)^n(c/p)^p\ldots$.

(4) Most 'Inequality' Theorems give maxima or minima results.

Note.—e. g. the minimum value of $(b+c)(c+a)(a+b)/abc$ is 8.

28. Theory of Numbers. [See also § 24 (5).]

(1) If $M \equiv N \pmod m$, $P \equiv Q \pmod m$,

$$M+P \equiv N+Q,\quad M-P \equiv N-Q,$$
$$MP \equiv NQ,\qquad M^k \equiv N^k \ (k \text{ integral}).$$

(2) The product of r consecutive integers is divisible by $r!$

Proof.—Induction, *or* use (6) to show that any prime is contained in $(n+r)!$ at least as often as in $n!\,r!$.

(3) *Fermat's Theorem.*

(*a*) $N^p - N \equiv 0 \pmod{p}$ if p be prime.

(*b*) $N^{p-1} - 1 \equiv 0 \pmod{p}$ p prime, N prime to p.

Proof.—$N, 2N, \ldots (p-1)N \equiv$ (not in order) $1, 2, \ldots (p-1) \pmod{p}$, &c.

(*c*) $N^{\phi(p)} - 1 \equiv 0 \pmod{p}$. N prime to p, $\phi(p)$ = no. int. $< p$ and prime to it.

(4) *Wilson's Theorem.* $(p-1)! + 1 \equiv 0 \pmod{p}$ if p be prime.

Note.—If p be not prime $(p-1)!+1$ is not divisible by p. One of the very few known tests for primes. Valueless, however, as such.

Proof.—The nos. $2, 3, \ldots (p-2)$ may be paired so that the product of each pair $\equiv 1$; $\therefore (p-1)! \equiv p-1$, &c.

(5) If $N = a^p b^q c^r \ldots$,

(*a*) The divisors of N are the various terms of

$$(1 + a + a^2 + \ldots + a^p)(1 + b + \ldots)(1 + c + \ldots)\ldots,$$

and this product represents their sum.

(*b*) The number of divisors is $(p+1)(q+1)\ldots$.

(*c*) The number of integers $< N$ and prime to it is

$$\phi(N) = N\Big(1 - \frac{1}{a}\Big)\Big(1 - \frac{1}{b}\Big)\ldots.$$

(6) The highest power of a prime p which divides $m!$ exactly is $\quad I(m/p) + I(m/p^2) + \ldots.$

Note.—$I(m/p^r) = I\{I(m/p^{r-1})/p\}$.

29. Convergence and Divergence of Series.

[*Definitions*: convergent, divergent,

absolutely convergent $(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots)$,

semi-convergent $(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots)$,

oscillating $(1 - 1 + 1 - 1 + \ldots)$.

Semi-convergent = conditionally convergent.]

(1) $S = 1/1^p + 1/2^p + 1/3^p + \ldots$ is C if $p > 1$, D if $p \not> 1$.

(2) $\left.\begin{aligned} S &= u_1 + u_2 + \ldots + u_n + \ldots \\ \Sigma &= v_1 + v_2 + \ldots + v_n + \ldots \end{aligned}\right\}$ (all terms +).

(*a*) If u_n/v_n is always finite and does not approach 0 in limit, S and Σ are C and D together.

(*b*) S is C when Σ is if $u_n/u_{n+1} > v_n/v_{n+1}$ always ;
" D " " " " $<$ " "

(3) $S = u_1 - u_2 + u_3 - \ldots$ is C if $u_1 > u_2 > u_3 \ldots$, provided $\mathrm{Lt}_\infty u_n = 0$. If $\mathrm{Lt}\, u_n \neq 0$, S is oscillating. This series may be semi-convergent (see *Note*).

(4) $S = u_1 + u_2 + \ldots + u_n + \ldots$	S is C.	S is D.	S is doubtful.
1. When $\mathrm{Lt}_{n=\infty}$ of u_n is		not 0	$= 0$
2. " " " nu_n is		not 0	$= 0$
3. " " " u_n/u_{n+1} is	> 1	< 1	$= 1$
4. " " " $n(u_n/u_{n+1} - 1)$ is	> 1	< 1	$= 1$

(5) $(1 + v_1)(1 + v_2)(1 + v_3) \ldots$ is finite if $v_1 + v_2 + v_3 + \ldots$ is C.

Note.—It is important to note that the terms of a series must not be regrouped unless the series be absolutely convergent.

30. Summation of Series. $[S = u_1 + u_2 + \ldots.]$

(1) *Method of Differences.*

$[u_2 - u_1 = \Delta u_1,\ \Delta u_2 - \Delta u_1 = \Delta . \Delta u_1 = \Delta^2 u_1,$ &c.$]$

(*a*) $u_n = (1 + \Delta)^{n-1} u_1$

$$= u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 u_1 + \ldots.$$

(*b*) $S_n = \left\{\frac{(1+\Delta)^n - 1}{\Delta}\right\} u_1 = nu_1 + \frac{n(n-1)}{2!}\Delta u_1 + \ldots.$

(*c*) *Montmort's Theorem.*

If $S = u_0 + u_1 x + u_2 x^2 + \ldots$ $[x < 1]$,

$$S_\infty = u_0 \cdot \frac{1}{1-x} + \Delta u_0 \cdot \frac{x}{(1-x)^2} + \Delta^2 u_0 \cdot \frac{x^2}{(1-x)^3} + \ldots.$$

S_n may be deduced from S_∞.

$$\text{Cor. } u_1 - u_2 + u_3 - \ldots = \frac{1}{2}u_1 - \frac{1}{2^2}\Delta u_1 + \frac{1}{2^3}\Delta^2 u_1 - \ldots.$$

Notes.—(1) Δ is an operation. Beware of supposing that

$$\Delta^2 u_1 = (\Delta u_1)^2$$

(2) The Δ's are best found by a tabular arrangement as shown. In the example given $u_1 = 1$, $\Delta u = 7$, $\Delta^2 u_1 = 12$, $\Delta^3 u_1 = 6$, and all higher Δ's vanish.

1	8	27	64	125
7	19	37	61	
12	18	24		
6	6			
0				

(3) The method leads to finite results only when u_n is of the form $An^p + Bn^{p-1} + \ldots$, and leads to no useful result when u_n contains terms involving n in the index (e. g. G. P.).

(4) If however u_n is known, it is best to express it as $a + bn + cn(n+1) + \ldots$, and use (5) below. This is better than to use (2) or (3) below.

(2) *Powers of Natural Numbers.* $S_r = 1^r + 2^r + \ldots + n^r$.

$$S_1 = \frac{n(n+1)}{2};\quad S_2 = \frac{n(n+1)(2n+1)}{6};\quad S_3 = \left\{\frac{n(n+1)}{2}\right\}^2 = S_1^2.$$

The following results are added for reference:

$$S_4 = \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1);$$
$$S_5 = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1);$$
$$S_6 = \tfrac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1);$$
$$S_7 = \tfrac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2);$$

and the following are interesting:

$$(S_4 - S_2)/(S_3 - S_1) = \tfrac{2}{5}(2n+1);\quad S_5 + S_7 = 2S_1^4.$$

[See also *DIFF. CALC.* § 7 (6) (*d*)].

(3) *Undetermined Coefficients.* Series to which (1) (*b*) is applicable may also be summed, if u_n is known, by assuming

$$S_n = a + bn + cn^2 + \ldots;$$

$\therefore\ u_n = S_{n+1} - S_n = b + c(2n+1) + \ldots$. Equate coefficients.

(4) If u_n can be split into the *difference* of two terms, one of which (v_{n-1}) is the same function of $n-1$ as the other (v_n) is of n, the sum is $v_n - v_0$.

Note.—One of the most general series coming under this head is

$$1 + \frac{p+r}{q+r} + \frac{(p+r)(p+2r)}{(q+r)(q+2r)} + \frac{(p+r)(p+2r)(p+3r)}{(q+r)(q+2r)(q+3r)} + \ldots.$$

The series referred to under (5) and (6) also come under this group. But in fact any series which can be summed is in this group, since

$$u_n = S_n - S_{n-1}.$$

(5) $u_n = (an+b)(a.\overline{n+1}+b) \ldots (a.\overline{n+m-1}+b)$.

Rule: To find the sum place one additional factor at the end of u_n, divide by the number of terms so increased, and by the common difference (a), and add an undetermined constant to the result, determining this constant by $n = 1$ or 2, &c.

Caution.—The first factors in successive terms must be part of the same A. P. as are the various factors in any term. Thus the rule cannot be applied to $u_n = n(n+2)(n+4)$. To evaluate this write

$$n(n+1)(n+2)+3n(n+1)+3n,$$

and use rule, or as n^3+6n^2+8n and use (2).

(6) $u_n = 1/(an+b)(a.\overline{n+1}+b) \ldots (a.\overline{n+m-1}+b)$.

Rule: To find the sum strike off the first factor, divide by the number of factors so diminished and by the common difference (a), *change the sign*, and add an undetermined constant, determining this as before.

Caution.—As in (5)

$u_n = 1/n(n+2)(n+4) = (n+1)(n+3)/n(n+1)(n+2)(n+3)(n+4)$, then write $(n+1)(n+3)$ as $n(n+1)+3n+3$. u_n is replaced by the sum of three terms to which the rule may be applied.

(7) $S = a+(a+b)x+(a+2b)x^2+\ldots$, one factor proceeding by A.P., the other by G.P. Multiply by x and subtract.

Note.—A particular case of (1) (c) above, also of (8). Many series, all of which are included in (1) (c), may be summed by a repetition of the operation.

(8) *Recurring Series.*

(a) If $2r$ terms are given, assume a G.F. of the type $(a+bx+cx^2+\ldots)/(1+px+qx^2+\ldots)$ with r terms in the numerator and $r+1$ in the denominator. Cross-multiply and equate coefficients.

(b) The G.F. is S_∞ when series is C.

(*c*) To find S_n split G.F. into partial fractions. These fractions allow u_n to be found.

(*a*) If there are no repeated factors in the denominator of the G.F. the partial fractions allow S_n to be written, for the part of S_n corresponding to the G.F. $A/(1-\alpha x)$ is $A(1-\alpha^n x^n)/(1-\alpha x)$.

(*β*) If there are repeated factors, find u_n and write:

$$S_n = u_0 + u_1 x + \ldots + u_n x^n,$$
$$px\,S_n = \quad pu_0 x + \ldots + pu_{n-1}x^n + pu_n x^{n+1},$$
$$qx^2 S_n = \quad \ldots \text{ and add to obtain } S_n.$$

(9) *Series derived by Differentiation, &c.* If

$$\phi(x) = a + bx + cx^2 + \ldots$$

can be summed to n terms or to ∞, then such series as

$$a + b\,.\,1^r\,.\,x + c\,.\,2^r\,.\,x^2 + \ldots$$
$$a + b\,.\,1^r\,.\,3^3\,x + c\,.\,2^r\,.\,4^3\,.\,x^2 + \ldots$$

can also be summed to n terms or to ∞, by repeated differentiations, &c.

Ex. $a+b\,.\,1^2x+c\,.\,2^2\,.\,x^2+\ldots = a+xD[xD\phi]$. $\{D = d/dx\}$.

Note.—While 'calculus-dodging' is not recommended it is of real help sometimes to notice that the results of differentiation may always be obtained by substituting $x+h$ for x on both sides of an equation and equating coefficients of x.

To a limited extent the integral calculus is available to sum series of the types $a+bx/1^r+cx^2/2^r+\ldots$, &c.

(10) *Binomial, Exponential, and Logarithmic Series.* [See §§ 19 (4), 22 (5).]

If $a, b, c, \ldots$ represent the coefficients of either the Binomial, the Exponential, or a Logarithmic Series, then the series

$$\frac{\phi(0)}{pq}a + \frac{\phi(1)}{(1+p)(1+q)}bx + \frac{\phi(2)}{(2+p)(2+q)}cx^2 + \ldots$$

can be summed if p and q are unequal positive integers ($q>p$), and $\phi(n)$ be a positive integral algebraical function of n, and the method is applicable when the denominators contain any, the same, number of unequal factors of the type. To start the summation:

(*a*) *Binomial and Exponential Series.* Complete the factorials which

already occur in the denominators of a, b, c... bringing them up in the successive terms to $q!$, $(q+1)!$, $(q+2)!$... by suitable multiplications for both numerator and denominator.

(*b*) *Logarithmic Series.* Split the coefficient $\phi(n)/(n+p)(n+q)$ into partial fractions.

(11) *Sum of selected terms of a known series.*

Ex. $\phi(x) = u_0 + u_1 x + u_2 x^2 + \ldots$

$\frac{1}{3}[\phi(x) + \phi(\omega x) + \phi(\omega^2 x)] = u_0 + u_3 x^3 + \ldots$, where $\omega^3 = 1$.

Note.—Similarly, using other complex roots of unity, for sum of other selected terms.

(12) *Sum of first $r+1$ coefficients of $\phi(x)$.*

If $\phi(x) = u_0 + u_1 x + u_2 x^2 + \ldots$,

$\phi(x)/(1-x) = u_0 + (u_0 + u_1)x + \ldots$,

and $u_0 + u_1 + u_2 + \ldots + u_r =$ coefficient of x^r in $\phi(x)/(1-x)$.

31. Linear Difference Equations.

(1) If $p_1 p_2 \ldots p_n$ are connected by the scale of relation $p_n + ap_{n-1} + bp_{n-2} = 0$, and α, β be roots of

$$x^2 + ax + b = 0,$$

then
$$p_n = A\alpha^n + B\beta^n \qquad (\alpha, \beta \text{ unequal})$$
$$= (A + Bn)\alpha^n \qquad (\alpha = \beta).$$

(2) If $p_n + ap_{n-1} + bp_{n-2} + cp_{n-3} = 0$, and α, β, γ are roots of $x^3 + ax^2 + bx + c = 0$,

$$p_n = A\alpha^n + B\beta^n + C\gamma^n \qquad (\alpha, \beta, \gamma \text{ unequal})$$
$$= (A + Bn)\alpha^n + C\gamma^n \qquad (\alpha = \beta)$$
$$= (A + Bn + Cn^2)\alpha^n \qquad (\alpha = \beta = \gamma).$$

32. Probabilities.

If an observed event has happened through some one of n mutually exclusive causes the probabilities of the existence of which, estimated before the event took place, are $P_1, P_2, \ldots P_n$, and if p_r denote the probability that when the rth cause exists the event will follow, the probability that the rth cause was the true cause of the observed event is

$$p_r P_r / \Sigma(pP).$$

THEORY OF EQUATIONS

[Throughout this section p's will be used for the coefficients of an equation when the binomial coefficients do not occur, a's when they do. Thus

$$(p_0 p_1 p_2 \dots p_n \text{ℒ} x\ 1)^n = p_0 x^n + p_1 x^{n-1} + \dots.$$

$$(a_0 a_1 a_2 \dots a_n \text{ℒ} x\ 1)^n = a_0 x^n + n a_1 x^{n-1} + \frac{n(n-1)}{2!} a_2 x^{n-2} + \dots].$$

1. The Horner 'Operation'.

If the coefficients of $f(x)$ be written from L. to R., with their proper signs, in a horizontal line, all missing terms being represented by zeros, then the Horner operation consists in multiplying the L. H. coefficient by a given quantity a, and adding the product to the term next on the R., multiplying the result by a and adding again to term next on R., and so on.

Ex. $f(x) = 2x^4 - 7x^2 - 5x + 1$. Operate with $a = 3$

2	0	−7	−5	1
	6	18	33	84
	6	11	28	85

or, better, combining multiplication and addition

2	0	−7	−5	1
	6	11	28	85

This operation gives:

(i) In the R. H. column the result of substituting $x = a$ in $f(x)$.

Ex. Thus 85 is the result of substituting $x = 3$ in $2x^4 - 7x^2 - 5x + 1$.

(ii) The result of dividing $f(x)$ by $x - a$.

Ex. Thus $2x^3 + 6x^2 + 11x + 28$, remainder 85, is the result of dividing $2x^4 - 7x^2 - 5x + 1$ by $x - 3$.

(iii) If the operation be repeated, stopping at the last

column but one, and again stopping at the last but two, and so on, the result of substituting $x+a$ for x and $f(x)$.

Ex.

2	0	−7	−5	1
	6	11	28	85
	12	47	169	
	18	101		
	24			

Thus $2x^4+24x^3+101x^2+169x+85$ is the result of substituting $x+3$ for x in $2x^4-7x^2-5x+1$.

2. General Properties of Equations.

(1) $f(x)$ has n roots, real or complex, and no more.

(2) $f(x) = (x-\alpha)(x-\beta)\ldots.$

Note.—$f(x) = p_0(x-\alpha)(x-\beta)\ldots$ if coefficient $x^n = p_0$.

(3) $\Sigma\alpha = -p_1/p_0$, $\Sigma\alpha\beta = p_2/p_0$, $\Sigma\alpha\beta\gamma = -p_3/p_0\ldots.$

Note.—$\Sigma\left(\frac{1}{\alpha}\right)$ from roots of $f\left(\frac{1}{x}\right) = 0$ or $= \frac{\Sigma\beta\gamma\delta\ldots}{\alpha\beta\gamma\delta\ldots}$.

Ex. If roots are in A. P., a, $a+d$, $a+2d$, ..., a and d can be calculated in terms of p_1 and p_2. Similarly if in G. P., H. P.

(4) *Complex Roots* enter in pairs, i.e. to every root $m+ni$ corresponds a root $m-ni$.

(5) If $f(h), f(k)$ have $\left.\begin{matrix}\text{same}\\ \text{contrary}\end{matrix}\right\}$ signs an $\left.\begin{matrix}\text{even or zero}\\ \text{odd}\end{matrix}\right\}$ number of roots lies between h, k.

(6) An equation of an odd degree has at least 1 real root.

(7) An equation of an even degree with last term − has at least 2 real roots of opposite signs.

(8) If the coefficients of $f(x)$ are all + there is no + root.

(9) If the coefficients of $f(x)$ are alternately + − and there is no term missing there is no − root.

(10) If $f(x)$, after the terms cease to be +, has all its remaining terms − it has one + root and only one.

Exx. $x^3-9x-2=0$, $x^5+x^3-7x^2-x-1=0$.

11) Descartes' Rule of Signs.

An equation $f(x) = 0$, whether complete or incomplete, cannot have more + roots than $f(x)$ has changes of sign, nor more − roots than $f(-x)$ has changes.

Ex. $x^5-5x-1 = 0$ cannot have more than 1 + root, for there is only one change in the series + − −, nor more than 2 − roots, therefore it must have at least 2 complex roots. It obviously has 1 + root and at least 1 − root, ∴ it must have a second − root or we should have an odd no. of complex roots. Hence the roots are 1+, 2−, 2 complex.

Note.—If all the roots are known real the no. of + roots is equal to the no. of changes, &c.

3. Transformation of Equations.

To form the equation with:

(*a*) Roots $-\alpha, -\beta, \ldots$. *Ans.* $f(-y) = 0$.

(*b*) Roots $m\alpha, m\beta, \ldots$. *Ans.* $f\left(\frac{y}{m}\right) = 0$.

(*c*) Roots $\alpha+h, \beta+h, \ldots$. *Ans.* $f(y-h) = 0$.

Common mistake to write $f(y+h) = 0$.

(*d*) Roots $\frac{1}{\alpha}, \frac{1}{\beta}, \ldots$. *Ans.* $f\left(\frac{1}{y}\right) = 0$.

(*e*) Roots $\alpha^2, \beta^2, \ldots$. Write $y = x^2$ in $f(x)f(-x) = 0$ after multiplication.

(*f*) Roots $\alpha^3, \beta^3, \ldots$. Write $y = x^3$ in

$$f(x)f(\omega x)f(\omega^2 x) = 0$$

after multiplication.

Note.—Arrange $f(x) = P+xQ+x^2R$ where P, Q, R contain x^3 only. The result of multiplication is $P^3+x^3Q^3+x^6R^3-3x^3PQR$.

(*g*) Roots $\phi(\alpha), \phi(\beta), \ldots$. Eliminate x between

$$f(x) = 0,\ y = \phi(x).$$

(*h*) Equation with any specified term removed. Use (*c*), and choose h to make specified term vanish.

(*i*) *Homographic Transformation.* $x = \frac{py+q}{p'y+q}$.

Note.—If $\alpha, \beta, \gamma, \delta$ be any 4 roots of either equation, $\frac{\alpha-\beta}{\beta-\gamma} \cdot \frac{\gamma-\delta}{\delta-\alpha}$ is unchanged by the transformation.

4. Depression of Equations.

If $\beta = \phi(\alpha)$ the degree of the equation can be depressed 2. For the equations $f(\alpha) = 0$, $f\{\phi(\alpha)\} = 0$ have a root in common. Find the H. C. F., hence α, hence β.

Note.—Special artifices often available as in Ex.

Ex. $x^4-2x^3-8x^2+23x-14 = 0$ has 2 roots whose sum is 3; solve. We must have $x^4-2x^3-8x^2+23x-14 = (x^2-3x+m)(x^2+x+n)$. Equate coefficients and solve. $m = 2$, $n = -7$.

5. Delimitation of Roots.

(1) *Superior Limit, Positive Roots.*

Method i. *Grouping.*

Ex. 1. $x^3(x^2-5x-13)+2x^2+x-70 = 0$.

Determine by mental solution of $x^2-5x-13 = 0$ the limit for the first bracket. This is $x = 7$. It makes the second bracket + also, and is the required limit.

Ex. 2. $x^5+2x^4+3x^3+8x^2-52x-80 = 0$.

Group as $x^5-80+2x(x^3-26)+x^2(3x-8) = 0$. Limit 3.

Ex. 3. $x^3-9x-2 = 0$. $x(x^2-10)+x-2 = 0$. Limit $x = \sqrt{10}$.

Method ii. If x^{n-r} be the highest power of x which has a negative coefficient, and p be the greatest negative coefficient, the limit is $1+\sqrt[r]{p}$.

Method iii. If each negative coefficient taken positively be divided by sum of all preceding + coefficients, the greatest of all the fractions thus formed, increased by 1, is the limit.

Proof.—Use the substitution $x^n = (x-1)(x^{n-1}+x^{n-2}+\ldots+1)+1$ in all + terms.

Notes.—(1) Methods ii and iii are rarely as serviceable as Method i.

(2) Better results will be obtained by both methods on putting $x = 2y$, or $x = 3y$ and obtaining the limits for y.

Method iv (**Newton**). Let the derived functions be $f_1(x)$, $f_2(x)$, $\ldots f_{n-1}(x)$. Determine the least value of x which makes $f_{n-1}+$. If this makes $f_{n-2}+$ test it on f_{n-3}. If not, increase it by 1 at a time until a value is reached which makes $f_{n-2}+$. Test this on f_{n-3}, increasing it if necessary. The

final value which makes all the f's including $f(x)$ + is the limit.

Method v (**Rolle's Theorem**). A real root of $f_1(x) = 0$ lies between every adjacent two of the real roots of $f(x) = 0$.

If the real roots of $f_1(x) = 0$ in descending order be $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$, then if the series of values $\infty, \alpha_1, \alpha_2, \ldots \alpha_{n-1}, -\infty$ be substituted for x in $f(x)$, the changes of sign in the results will determine the number and position of *all* the real roots of $f(x) = 0$. No more than one real root of $f(x) = 0$ can lie between two adjacent roots of $f_1(x) = 0$. If $f_r(x) = 0$ has k imaginary roots $f(x) = 0$ has at least k.

Ex. Hence $x^5 + 5x + p = 0$ has only one real root.

Method vi (**Waring**). Form the equation whose roots are the squares of the differences of the roots of $f(x) = 0$. [See § 11 (1), (4).] Let δ be the inferior limit of its + roots. Take $k = \sqrt{\delta}$, and let s = superior limit for $f(x)$. If the series of values $s, s-k, s-2k, \ldots$ be substituted for x in $f(x)$, the changes of sign in the result will detect the number and position of *all* the real roots of $f(x) = 0$.

Theoretically perfect, but practically useless except for cubics. k as thus determined usually much too small.

(2) *Inferior Limit, Positive Roots.* Put $x = \frac{1}{y}$ and determine superior limit for y.

(3) *Negative Roots.* Put $x = -y$ and determine + limits for y.

(4) *Fourier's Theorem.* The number of real roots of $f(x) = 0$ which lie between α and β $(\beta > \alpha)$ is not greater than the excess of the number of changes of sign in the series $f(\alpha), f_1(\alpha), f_2(\alpha), \ldots f_n(\alpha)$ over the number of changes in the series $f(\beta), f_1(\beta), f_2(\beta), \ldots f_n(\beta)$ and differs from this excess always by an even number.

Note.—Descartes' Rule [§ 2 (11)] is a particular case.

Budan's Statement. Write $x + \alpha$ for x in $f(x)$ [see § 1 (iii)], and count the number of changes of sign in the result.

Similarly write $x+\beta$ and count. The number of real roots between α and β is either the difference of these two counts or differs from it by an even number.

(5) *Sturm's Theorem.* Let $f(x)=0$ be an equation cleared of equal roots. [See § 6.]

Perform the operation of finding the H.C.F. of $f(x)$ and $f_1(x)$, but change the sign of every final remainder *before* it is used as a divisor. Call these modified remainders $F_2(x)$, $F_3(x)$, ...

Let the last remainder, also with its sign changed, be $F_m(x)$.

Then the number of real roots between α and β ($\beta > \alpha$) is *equal* to the excess of the number of changes of sign in the series $f(\alpha), f_1(\alpha), F_2(\alpha), \ldots F_m(\alpha)$, over the number of changes of sign in the series

$$f(\beta), f_1(\beta), F_2(\beta), \ldots F_m(\beta).$$

Notes.—(1) F_m is independent of x and cannot be zero unless equal roots are present.

(2) If equal roots are present the theorem gives the no. of distinct roots.

(3) If in the process of finding the F's we reach one, F_r, which cannot change sign [e. g. $3x^2+5$] we can stop the process at this point and apply the theorem to $f, f_1, F_2, \ldots F_r$.

(4) In applying the H. C. F. process we may multiply or divide either divisor or dividend by any *positive* quantity, and this multiplication may take place at any stage during the process of division provided that *it is applied to the whole of what is left* of the dividend.

(5) We rarely need calculate F_m, for since a value which causes F_{m-1} to vanish will give opposite signs to F_{m-2} and F_m we can determine its sign by inspection.

Proof.—One change is lost between f and f_1 when x passes through a root. No change is lost in the series when x passes through a root of F_r, because F_{r-1}, F_{r+1} have opposite signs if $F_r = 0$.

Ex. $f(x) = x^5-5x-1$. f_1 (rejecting 5) $= x^4-1$. $F_2 = 4x+1$. $F_3 = +$ [Note (5)]. Hence the changes are :

x	f	f_1	F_2	F_3		
∞	+	+	+	+	change	0
2	+	+	+	+	changes	0
1	−	0	+	+	,,	1
0	−	−	+	+	,,	1
-1	+	0	−	+	,,	2
-2	−	+	−	+	,,	3
$-\infty$	−	+	−	+	,,	3

Hence the only real roots lie between 2, 1 ; 0, -1 ; -1, -2.

6. Equal Roots.

The equal roots of the equation $f(x) = 0$ are given by the various factors of the H. C. F. of $f(x)$ and $f_1(x)$ each repeated once more frequently than in this H. C. F.

7. Commensurable Roots (Newton).

Calculate $f(1), f(-1)$. If either of these vanish, divide by $x-1$ or $x+1$, noting roots 1 or -1.

Make coefficient of highest term unity. [Substitute $x = ky$.]

Determine possible limits for roots of resulting equation.

Write down in a horizontal line all divisors (except 1, -1) of absolute term which lie between these limits.

Note.—If a is a root, $a-1$ divides $f(1)$, $a+1$ divides $f(-1)$. Reject any divisors which do not satisfy this condition.

Divide the absolute term by each of these divisors, adding to each quotient the last coefficient but one, and writing the result for each divisor immediately below it.

If any number results which is not divisible by the number at the head of the column, reject that column.

Divide those numbers which are divisible and add to quotient the next coefficient on L. H. Again reject or divide, and so on.

If finally, after $n-1$ divisions, the quotient -1 is obtained, the number at the head of the column is a root.

Notes.—(1) If the coefficient of the highest term be 1 there are no fractional commensurable roots.

(2) If the equation cleared of commensurable roots thus obtained

have an absolute term which is divisible by any one of them, it must be tested to see whether such root be repeated.

Ex. $x^4-x^3-x^2+19x-42=0$. Limits for roots 3, -5 (Method of Grouping): $f(1)=-24, f(-1)=-60$.

Divisors of -42: 2, -2, -3, -4. Of these -4 is rejected by $f(1)$ test.

2	-2	-3
-2	40	33
-2	-21	-12
-2	*	3
-1		-1

Roots 2, -3.
Equation $(x-2)(x+3)(x^2-2x+7)=0$.

8. Numerical Evaluation of Roots.

[Of these methods i and ii are most generally useful. i can be relied on to 2 significant figures, or, with very careful drawing, to 3, and may be supplemented by iii when greater accuracy is required. Method iii is to be recommended as more speedy than v if 7-fig. accuracy will suffice, and is rather less subject to error. Method iv is novel, as far as English textbooks are concerned, and will often give 3 and even 4 figures with much less mental work than any other. Method vi is little known. Method ix deserves more attention than it has received from English textbooks. Method x is almost useless.]

Method i (*Graphical*).

Ex. 1. $x^3-9x-2=0$. Trace the curves $y=x^3$, $y=9x+2$. *Or*, Trace the curves $y=x^2$, $(y-9)x=2$. Their intersections give all the real roots.

Note.—The curve $y=x^3$ can be obtained in celluloid [Messrs. Griffin], as also other curves.

Ex. 2. $\sin x-x\cos x=0$. Trace the curves $y=x$, $y=\tan x$.

Note.—The general quartic can be graphically solved by the intersections of the parabola $y=x^2+2ax+p$ and the circle

$$x^2+y^2+2qx+r=0.$$

Method ii (*Interpolation*). The rule is similar to that employed in the use of logarithms and trigonometrical tables. If α, β $(>\alpha)$ be two values of x between which a root of $f(x)$

lies, then, if $f(\alpha) = A, f(\beta) = -B$, an approximation to the value of x will be

$$\alpha + \frac{A}{A+B}(\beta - \alpha) = \frac{A\beta + B\alpha}{A+B}.$$

Notes.—(1) This method is sometimes called the *Regula Falsi.*

(2) It is particularly adapted to exponential and similar equations, though it may be applied to any.

(3) For a further example of it see Method xii.

Ex. Solve $x^x = 3$. It is obvious that a root lies between 1 and 2, nearer 2. Write the equation $\log x + \log\log x = \log\log 3 = \bar{1}{\cdot}6786$ and tabulate L.H. for the values 1·5, 1·6, 1·7, 1·8, 1·9. Applying interpolation to this table we obtain as an approximate value of x 1·825. Again using 4-fig. logs tabulate for values 1·824, 1·825, 1·826. We get 1·8254. Now using 7-fig. logs tabulate for 1·8253, 1·8254, 1·8255 and we obtain 1·825455.

Method iii (**Newton**). If c be an approximate root of $f(x) = 0$, then $c - f(c)/f_1(c)$ is usually a better approximation.

Ex. $x^3 - 9x - 2 = 0$. The successive approximations to one root are: 3, 3·1, 3·105, 3·10548 ....

Notes.—(1) Will usually succeed unless there are *two* roots near c.

(2) (**Fourier**) Will not, however, succeed with certainty unless for the values over which the approximation is proceeding $f(x)/f_1(x)$ is $-$, and also $f(x)/f_2(x)$ is $+$.

(3) Particularly valuable for trigonometrical equations, &c.

Ex. $\sin x - x\cos x = 0$. The approximations to one root are: $4{\cdot}5\,\pi$, $4{\cdot}477\,\pi$, $4{\cdot}47741\,\pi$, ... *Method* iv can also be applied to $x = \tan^{-1} x$.

(4) Applicable to two or more simultaneous equations: $f(x, y) = 0$, $\phi(x, y) = 0$. If ξ, η be approximate solutions to these, then $\xi + h$, $\eta + k$ will generally be closer approximations where h, k are given by

$$f(\xi, \eta) + hf_\xi + kf_\eta = 0$$
$$\phi(\xi, \eta) + h\phi_\xi + k\phi_\eta = 0.$$

Ex. $x^3 - y^3 - 25 = 0$, $x + y - 4 = 0$ } Approximations: x : 3, 2·97, 2·9664, y : 1, 1·03, 1·0336, &c.

Method iv (**Ross**). The following method is exceedingly elegant, and if Barlow's Tables are available, as they ought to be, is often the shortest and safest method of finding 4 significant figures of all the real roots, especially of trinomials.

Ex. $x^3-9x-2=0$. (*a*) Write $x=\sqrt[3]{9x+2}$. Start with *any* value of x above the limit of positive roots, preferably the actual limit; substitute in R.H.; evaluate by Barlow; substitute again, &c. The values thus obtained converge to the greatest root. Tabulate thus:

x	$9x+2$	$\sqrt[3]{9x+2}$	
4	38	3·36	
3·36	32·24	3·18	[Use tables for 3180^3]
3·18	30·62	3·128	
3·128	30·152	3·112	
3·112	30·008	3·108	
3·108	29·972	3·106	
3·106	29·954	3·1056	[Mental interpolation]
3·1056	29·9504	3·1055	

(*b*) For the next root take x a little lower than the first root and substitute in $x=(x^3-2)/9$. Thus starting with $x=3$ the successive approximations are 2·8, 2·2, ·95, −·13, −·222, −·223, −·2235.

(*c*) For the next root take a number a little less than the root just found and use $x=\sqrt[3]{9x+2}$ again. The successive approximations are $x=-·23$, $-·41$, $-1·19$, $-2·06$, $-2·55$, $-2·75$, $-2·83$, $-2·86$, $-2·874$, $-2·881$, $-2·8816$, $-2·8818$, ...

Notes.—(1) The first four or five steps should be done with rough accuracy mentally until the numbers begin to converge.

(2) The method has the curious property that numerical mistakes will not usually prevent eventual success, though they may transfer the evaluation from one root to another.

(3) Seven-figure accuracy may be obtained by the use of logarithms, but if this is desired it is better to apply Newton's Method (iii) when 4 figures are obtained.

(4) The method may be applied in the form $x=\sqrt{9+\frac{2}{x}}$. Thus successive approximations are 3, 3·11, 3·1053, ...

(5) An equation such as $x^5=5x-1$ can be treated as $x^3=\frac{5}{x}-\frac{1}{x^2}$. The successive values are 2, 1·3, 1·48, 1·430, 1·443, 1·440,... the correct value being 1·44050....

(6) It is applicable to exponential equations, &c. Thus $e^x=2x^2$ is treated as $x=\log_e 2+2\log_e x$, and starting with the value $x=3$, a rather slow convergence produces the value 2·62, which Newton's Method changes into 2·61787....

(7) No preliminary separation of the roots is required. Even the delimitation of them is not necessary.

(8) The method appears to have been first given in English, at all events with any complete explanation, by Ross (*Nature*, vol. 78, p. 663), but it is substantially identical with the method which German text-books call *Iteration* (Runge, *Praxis der Gleichungen*, pp. 56–9, 81 ff.).

(9) It may be applied to simultaneous equations, and where applicable is often easier than Method iii.

Ex. $\begin{matrix} x^3-y^3-25 = 0 \\ x+y-4 = 0 \end{matrix}$. Write $x = \sqrt[3]{y^3+25}$, $y = 4-x$. $y = 1$ is approximate solution. This gives $x = \sqrt[3]{26} = 2{\cdot}96$. This (second equation) $y = 1{\cdot}04$. This (first equation) $x = 2{\cdot}967$. Hence successively $y = 1{\cdot}033$, $x = 2{\cdot}9664$, $y = 1{\cdot}0336$, and, using 7-fig. tables, $x = 2{\cdot}96645$, $y = 1{\cdot}03355$, $x = 2{\cdot}966444$, $y = 1{\cdot}033556$, $x = 2{\cdot}966445$, $y = 1{\cdot}033555$, which figures are probably correct.

Method v (**Horner**). Determine by trial, or by § 5 (5), two integers between which the root lies, or, if a decimal, its first figure.

Write the equation with coefficients only, and with missing terms shown by zeros.

'Operate' [§ 1] across with the first figure (or integral part) of root.

Operate again, stopping at the last column but one.

Operate again, stopping at the last column but two, and so on until the operation is performed on one column only.

Affix 0 to second column on L.H., 00 to next, 000 to next, and so on.

An estimate of the next figure of the root may now be obtained by dividing the figure which now stands in the last column by the figure in the last but one, making a rough calculation as to the effect upon this figure (in the last but one) of the operations performed by the quotient, and choosing, usually, the highest integer which after the operations will not result *in a change of sign in the last term.*

Repeat the series of operations with this integer, again affix 0's, and so on.

After 3 or 4 figures have been obtained we may obtain several others correctly by the following approximation:

Instead of adding 0's cut off one fig. from R.H. of last column but one, two from last but two, &c., and repeat the series of operations. Then again abbreviate until all the columns have been cut away.

Ex. $x^3-9x-2=0$.

1	0	−9	−2	3·1054826167
	3	0	−2000	
	6	1800	− 109000000	
	90	1891	− 9617375	
	91	19830000	− 1646659	
	92	19876525		
	9300	19923075	The process is now completed by the contracted division of 199313 into 1646659. The last figure thus obtained is inaccurate.	
	9305	1992307		
	9310	1992679		
	9315	1993051		
0	93	199305		
	1...	199313		

The roots of this equation are:

3·1054826165..., −·2234620717..., −2·8820205448....

Notes.—(1) The case where 2 roots are nearly equal requires care until they are separated. Until this point is passed a better estimate of the next figure is given by *twice* the last column divided by the last but one. Or use Method iv or vii (see *Note*).

(2) If the root which is being evaluated is less than 1, use § 3 (*b*) to multiply the roots by such a figure as to bring it between 1 and 10. Thus to evaluate the second root of the above equation, put $y=-10x$, i. e. find the root of $y^3-900y+2000=0$ which lies between 2 and 3.

Method vi (**Weddle**). The method will be explained in its application to $x^3-9x-2=0$. It is, of course, perfectly general. Determine two integers between which a root lies, 3 and 4. Put $x=3y$. The equation in y, $f(y)=0$, has a root between 1 and 2. An approximation to this root may be obtained by putting $y=1+c$; $\therefore f(1+c)=0$; $\therefore f(1)+cf'(1)=0$ if c is small; $\therefore c=-f(1)/f'(1)$. In our case $c=\cdot03$. Put $y=1\cdot03z$, and determine z in the same way, 1·005. Put $z=1\cdot005w$ and determine w, &c. Finally the root is $3\times1\cdot03\times1\cdot005\times1\cdot00001\times1\cdot0000005$.

The whole work for the determination of 7 figures is exhibited below:

x^3	$-9x$	$f(1)$	$f'(1)$	'Radix'
				3
27·		27 $\times 3 =$	81	
·81	-27	-27 $\times 1 =$	-27	
27·81	·81	-2		
·8343				
28·6443		-2	54	1·03 $[=1+\frac{2}{54}$
·859329				
29·503629		29·503629 $\times 3 =$	89	
147518				
29·651147	$-27·81$	$-27·81$ $\times 1 =$	-28	
148256	·13905	-2		
29·799403		$-$ ·306371	61	1·005 $[=1+\frac{·306}{61}$
148997				
29·948400		29·948400 $\times 3 =$	90	
299				
29·948699	$-27·94905$	$-27·94905$ $\times 1 =$	-28	
299	279	-2		
29·948998		$-$ ·000650	62	1·00001
299				
29·949297		29·949297 $\times 3 =$	90	
	$-27·949329$	$-27·949329 \times 1 =$	28	
		-2		
		$-$ ·000032	62	1·0000005

100
1·0000105
50001
1·0050106
301503
1·0351609
3·1054827

Explanation. The dotted lines are inserted to divide the work into stages and render it easier to follow. After putting $x = 3y$, thus using 'radix' 3, the equation becomes $27x^3 - 27x - 2 = 0$, and the work in the top space shows the evaluation of $f(1)$ and of $f'(1)$, namely $27 \times 3 - 27 \times 1$, and the deduction of the second radix 1·03. The work in the next space shows the formation of the new equation $27(1·03)^3 x^3 - 27(1·03)x - 2 = 0$, the first column giving the successive multiplications of 27 by 1·03 three times, the second giving the second term in the

equation, the third giving $f(1)$, and the fourth $f'(1)$, leading to the next radix 1·005. Finally at the right of this explanation the five radices are multiplied together to give the root 3·1054827. The successive forms assumed by the equation are shown by the figures in italics.

Notes.—(1) This method, which has been strangely neglected, is due to T. Weddle (R. S. *Trans.*, Ap. 29, 1841).

(2) If Compound Interest Tables are available, the work in the first column may be dispensed with and the method is exceedingly brief. In any case it is less subject to error than most.

Method vii (**Lagrange**). Determine a the integer next below a root of $f(x) = 0$. Substitute $a + 1/y$ for x. If there is only one root between a and $a+1$, the resulting equation in y will have only one positive root (> 1). Let b be the integer next below this root. Substitute $y = b + 1/z$ and so on. The value of x is $a + \cfrac{1}{b+} \cfrac{1}{c+} \ldots$.

Ex. $x^3 - 9x - 2 = 0$. One root is between 3 and 4.

1	0	−9	−2	
	3	0	−2	
	6	18		→
	9			Operator 3

The equation in y is therefore $-2y^3 + 18y^2 + 9y + 1 = 0$, and the Horner operation is applied to this after finding a root between 9 and 10. Thus the operation is worked alternately from L. to R., and from R. to L. The safest arrangement for the work is as follows :

1	9	18	−2	
82	9	0		←
	−153	−18		Operator 9
		−36		
82	−153	−36	−2	
	11	−14	−30	→
	175	336		Operator 2
	339			
82	339	336	−30	

and so on. The next operator is $\overleftarrow{12}$ and the root is given in the form

$$3 + \cfrac{1}{9+} \cfrac{1}{2+} \cfrac{1}{12+} \cfrac{1}{6+} \cfrac{1}{1+} \cfrac{1}{2+} \cfrac{1}{1+} \cfrac{1}{1+} \cfrac{1}{1+} \cfrac{1}{11+} \ldots$$

On a rough average each convergent involves accuracy to a fresh decimal place. The value of the fraction written is 3·10548261657....

Note.—The method is more risky numerically than Horner's, but has the great advantage that separation of the roots is not required. If one of the subsidiary equations has more than one positive root, the work may be continued with either. If a subsidiary equation repeats itself, the root may be completely determined [see *ALG* § 23 ii (1)].

The process may at any stage be continued by Horner's Method or may replace Horner's Method with advantage until separation of the roots has been effected.

Method viii (**Lagrange**).

Ex. $x^3-9x-2=0$.

Let $x=a$ be a root, then $x^3-9x-2=(x-a)(x^2+px+q)$;

$$\therefore \quad \frac{x^3-2}{x}-9=\left(1-\frac{a}{x}\right)(q+px+x^2).$$

Since $\log(q+px+x^2)$ contains only + indices of x,

$$-a=\text{co.}\,\frac{1}{x}\ \text{in}\ \log\left[1-\frac{1}{9}\left(\frac{x^3-2}{x}\right)\right];$$

$$\therefore \quad a=\text{co.}\,\frac{1}{x}\ \text{in}\ \frac{1}{9}\left(\frac{x^3-2}{x}\right)+\frac{1}{2}\cdot\frac{1}{9^2}\left(\frac{x^3-2}{x}\right)^2+\frac{1}{3}\cdot\frac{1}{9^3}\left(\frac{x^3-2}{x}\right)^3+\ldots;$$

$$\therefore \quad a=-\frac{2}{9}-\frac{2^3}{9^4}-3\frac{2^5}{9^7}-12\frac{2^7}{9^{10}}-\ldots,$$ a very rapidly converging series giving $a=-\cdot 223462\ldots$.

Notes.—(1) This method is applicable only to equations of the type $x^n+cx-b=0$, and then only produces a rapidly convergent series if $c>b$ and $c>1$. The solution in this case is

$$x=\frac{b}{c}-\frac{b^n}{c^{n+1}}+\frac{2n}{2}\frac{b^{2n-1}}{c^{2n+1}}-\frac{3n(3n-1)}{2\cdot 3}\frac{b^{3n-2}}{c^{3n+1}}+\ldots.$$

(2) It gives the *numerically least* root of the equation and *this only*.

(3) A similar method applied to $x^{2n+1}+cx^2-b=0$ gives $\alpha+\beta$ and $\alpha^2+\beta^2$ and thus α and β where α and β are the two numerically least roots.

(4) The theorem in the Differential Calculus known as Lagrange's can often be applied to the solution of equations by series. See *DIFF. CALC.* § 6 (8).

Method ix [**Gräffe**]. (*a*) Let $f=0$ be the equation. Form the equation, $f^2=0$, whose roots are the squares of the roots of the given equation [§ 3 (*e*)]. Repeat the operation, f^4, and so on. Unless the roots are nearly equal in value numerically (and even then eventually), one or more of the

terms at R.H. of equation will soon become small compared with the terms immediately preceding. When this occurs, say with f^r, the equation formed by excluding these smaller terms will give an approximation to the rth powers of the larger roots, that obtained by the smaller terms together with the *last* of the larger terms to the rth powers of the smaller roots.

(*b*) Further, if $f^r = p_0x^n - p_1x^{n-1} + p_2x^{n-2} - \ldots$, and if the roots are all real, approximations to the rth powers of all the roots in descending order are given by p_1/p_0, p_2/p_1, $p_3/p_2, \ldots$.

Proof.—(*a*) Consider $(x-p)(x-q)(x-r) = 0$. If p and q be large compared with r this is approximately $x[x^2-(p+q)x+pq]-pqr = 0$, &c.

(*b*) If $\alpha, \beta, \gamma, \ldots$ be the roots of f^r in descending order, $\frac{p_1}{p_0} = \Sigma\alpha = \alpha$ nearly, $\frac{p_2}{p_0} = \Sigma\alpha\beta = \alpha\beta$ nearly, $\therefore \frac{p_2}{p_1} = \beta$ nearly.

Ex. $f = x^3-9x-2 = 0$. The coefficients are

f^1	1	0	−9	−2
f^2	1	−18	81	−4
f^4	1	−162	6417	−16
f^8	1	−13410	41172705	−256

At this stage it is easy to see that the fourth column will not affect the first five figures of the third column in f^{16}. The roots are therefore separated and correct to 5 figures, the eighth power of the smallest root is given by $41172705x-256 = 0$, while to the same accuracy the eighth power of the other two roots is given by

$$x^2-13410x+41172705 = 0.$$

Notes.—(1) For fuller details see *Encycl. Math.*, Bd. i. 1, pp. 440 ff., or Runge, *Praxis der Gleichungen*, pp. 157–82.

(2) If *all* the roots of an equation are required, this method is preferred by German writers as the quickest. It is difficult, however, to tell beforehand to what equations it is suitable. If the roots are numerically close together, or if the imaginary roots have a modulus near in value to the real roots, a high f will be required before the roots are separated, and the labour of finding this is great unless a calculating machine (for which the method is eminently suitable) is available. The method may be tried in a few minutes up to f^8 and

will indicate with great exactness any root materially smaller or larger than all the rest.

(3) If the work is taken beyond f^8, only 7 figures, at most, should be retained, and the notation $4{\cdot}117270_7$ may be employed to indicate $4{\cdot}117270 \times 10^7$. This notation has the advantage of showing easily when one term ceases to affect the others.

(4) In the writer's experience the series $f^3, f^9, f^{27}\ldots$ is to be preferred to Gräffe's series for equations of degree not higher than 5. It is almost as easy to calculate [§ 3(f)], more rapidly effective, and has the great advantage of not obscuring the sign of the roots.

(5) While great labour is often required to separate the roots for the (b) method, the (a) method will often suffice. Thus in the case given f^3 can be found mentally, and all three roots from it, correct to 4 places, if Barlow's Tables are available, in *less than one minute*.

Method x [**Bernoulli**]. If s_p denotes the sum of the pth powers of the roots [see § 11 (2)], the series of numbers s_2/s_1, s_3/s_2, $s_4/s_3,\ldots$ will usually converge towards the value of the numerically greatest root.

Note.—This method is always tedious, and valueless in practice if a second root is *numerically* near in value to the greatest root (as e.g. in $x^3-9x-2=0$), or if there be a modulus of imaginary roots near in value to the greatest root. If such modulus be greater than the greatest root, the fractions have no limiting value.

Method xi [For *Cubic Equations*]. To solve $x^3+px+q=0$ in the case when all the roots are real, put $x = ky$,

$$\therefore \quad k^3y^3+pky+q=0.$$

But $\cos^3\alpha - \frac{3}{4}\cos\alpha - \frac{1}{4}\cos^3\alpha = 0$. Hence the equation is satisfied by $y = \cos\alpha$ if

$$\frac{p}{k^2} = -\frac{3}{4}, \quad \frac{q}{k^3} = -\frac{1}{4}\cos^3\alpha.$$

Eliminate k. Determine $\cos^3\alpha$, three values for $\cos\alpha$, for y, for x.

Note.—Of little practical value. A similar method involves $\cosh\alpha$.

Method xii [*Complex Roots*].

Ex. $x^5-5x-1=0$. (a) Here the real roots are $-{\cdot}200064$ [best by Lagrange's series, or Gräffe], $-1{\cdot}440501$, $+1{\cdot}541652$. The sum of these is $-{\cdot}098913$. But the sum of all the roots is 0, $\therefore$ if

$$r(\cos\phi \pm i\sin\phi)$$

be the two complex roots $2r\cos\phi = \cdot 098913$. But the product of all the roots is 1, $\therefore r^2 = 2\cdot 250769$. Hence r, and ϕ.

(*b*) The following method does not require the real roots to be known and is (theoretically) available when there are two or more pairs of complex roots. Put $x = r(\cos\phi + i\sin\phi)$, equate real and imaginary parts, and obtain

$$r = -\frac{\sin 5\phi}{5\sin 4\phi} \text{ and } (\tfrac{1}{5}\sin 5\phi)^5 - \sin^4 4\phi \sin\phi = 0 = F(\phi), \text{ say.}$$

A little consideration will show that there is a value of ϕ near 90°. Method ii is now employed. By the aid of 4-fig. log. tables calculate $F(89°)$, $F(88°)$, $F(87°)$, $F(86°)$. The calculation is not tedious if the four F's are evaluated *simultaneously*. Plot the values on squared paper. A value slightly above 88° is indicated. Again with 4-fig. tables calculate F for 87° 54′, 88°, 88° 6′, 88° 12′. A value near 88° 6′ is indicated. With 7-fig. tables plot F for 88° 6′, 88° 6′ 20″, 88° 6′ 40″, 88° 7′, and we obtain $\phi = 88°\ 6'\ 39\cdot 2''$, whence the complex roots are

$$\cdot 049455\ldots \pm (1\cdot 499442\ldots)i$$

Note.—Gräffe's Method can also be used to find the modulus (r) of the complex roots, provided that this is not close in value to one of the real roots.

9. Binomial Equations.

(1) If α is a complex root of $x^n - 1 = 0$, α^m will also be a root (m integral).

(2) Further, if n is prime, the whole series of roots is $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$.

(3) If m, n are prime to each other, $x^m - 1 = 0, x^n - 1 = 0$ have no common root except 1.

(4) If m, n have H. C. F. k, the roots common to $x^m - 1 = 0$, $x^n - 1 = 0$ are all roots of $x^k - 1 = 0$.

Note.—(3) is a particular case of (4).

(5) If $n = p^a q^b r^c \ldots$ ($p, q, r \ldots$ all primes), and α is any root of $x^{p^a} - 1 = 0$, then $\alpha, \beta, \gamma \ldots$ all satisfy $x^n - 1 = 0$, and all the roots of $x^n - 1 = 0$ are n products of the type $\alpha\beta\gamma \ldots$.

(6) *Special Roots* (*Primitive Roots*).

(*a*) If n is a prime, all the complex roots of $x^n - 1 = 0$ are 'special'.

(*b*) If $n = p^a$, the special roots are determined by

$$(x^n - 1)/(x^{n/p} - 1) = 0.$$

(*c*) If $n = p^a q^b r^c \ldots$, the special roots are given by

$$\frac{x^n - 1}{\Pi(x^{n/p} - 1)} \cdot \frac{\Pi(x^{n/pq} - 1)}{\Pi(x^{n/pqr} - 1)} \cdots = 0;$$

and there are $n\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)\ldots$ of them.

Ex. $x^{60} - 1 = 0$, $60 = 2^2 . 3 . 5$. There are $60\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 16$ special roots, determined by

$$\frac{(x^{60} - 1)(x^{10} - 1)(x^6 - 1)(x^4 - 1)}{(x^{30} - 1)(x^{20} - 1)(x^{12} - 1)(x^2 - 1)} = 0.$$

(7) The roots of binomial equations can all be completely determined by De Moivre's Theorem, and are

$$x = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n} \quad [k = 0, 1, \ldots \overline{n-1}].$$

(8) *Gauss's Equation*, $x^{17} - 1 = 0$. The roots of this equation can be completely determined by quadratic equations only. Let α be any complex root. All the complex roots are included in $\alpha, \alpha^2, \ldots \alpha^{16}$ [see (2)]. Divide these into two groups

$$\theta_1 = \alpha + \alpha^2 + \alpha^4 + \alpha^8 + \alpha^{16} + \alpha^{15} + \alpha^{13} + \alpha^9$$
$$\theta_2 = \alpha^3 + \alpha^6 + \alpha^{12} + \alpha^7 + \alpha^{14} + \alpha^{11} + \alpha^5 + \alpha^{10}.$$

These groups are easily formed if it is remembered that each term is the square of the preceding and that $\alpha^{17} = 1$. Prove θ_1, θ_2 roots of

$$\theta^2 + \theta - 4 = 0.$$

By taking alternate terms divide the θ's into two groups

$$\theta_{11} = \alpha + \alpha^4 + \alpha^{16} + \alpha^{13}; \quad \theta_{21} = \alpha^3 + \alpha^{12} + \alpha^{14} + \alpha^5;$$
$$\theta_{12} = \alpha^2 + \alpha^8 + \alpha^{15} + \alpha^9; \quad \theta_{22} = \alpha^6 + \alpha^7 + \alpha^{11} + \alpha^{10}.$$

Prove θ_{11}, θ_{12} roots of $\theta^2 - \theta_1\theta - 1 = 0$; θ_{21}, θ_{22} roots of $\theta^2 - \theta_2\theta - 1 = 0$. Again divide θ_{11} into groups and prove θ_{111}, θ_{112} roots of

$$\theta^2 - \theta_{11} . \theta - \theta_{21} = 0.$$

Finally α, α^{16} are roots of $\theta^2 - \theta_{111} . \theta + 1 = 0$.

Further details on this problem, which Gauss considered his most remarkable discovery, and various methods of construction of the regular 17-gon geometrically in Enriques, *Fragen der Elementargeometrie*, ii. 171–88.

10. Reciprocal Equations. These are of two classes:

i. Coefficients of terms equidistant from the end equal and same sign. Root -1 if degree is odd.

ii. Coefficients of terms equidistant from the end equal and opposite sign. Root $+1$ if degree is odd, ± 1 if even.

On removing factors $x+1$, $x-1$, or x^2-1, we have in every case a R. E. of class i, degree even $(2m)$. The degree can now be halved by dividing throughout by x^m and using the substitution

$$y = x + \frac{1}{x}, \quad y^2 - 2 = x^2 + \frac{1}{x^2}, \text{ \&c.}$$

Notes.—(1) Other equations can sometimes be reduced to the reciprocal form, e.g. $8x^3+20x^2+10x+1 = 0$: put $2x = y$.

(2) The successive values for $x^3 + \frac{1}{x^3} = t_3$ may be calculated from the relation $t_p = t_1 t_{p-1} - t_{p-2}$; thus $t_3 = y(y^2-2) - y = y^3 - 3y$, &c.

11. Symmetric Functions.

(1) *Method by Subsidiary Equations.* Illustrated by cubic.

(*a*) Functions of $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\alpha+\beta}$;

$$y = \frac{\alpha}{\beta+\gamma} = \frac{\alpha}{\alpha+\beta+\gamma-\alpha} = -\frac{p_0\alpha}{p_1+p_0\alpha}.$$

Eliminate α between this and $(p_0, p_1, p_2, p_3)(\alpha, 1)^3 = 0$.

(*b*) Functions of $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$;

$$y = (\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma$$

$$= (\alpha+\beta+\gamma-\alpha)^2 - \frac{4\alpha\beta\gamma}{\alpha} = \left(\frac{p_1+p_0\alpha}{p_0}\right)^2 + \frac{4p_3}{p_0\alpha}.$$

Note.—Best, however, by first removing the x^2-term of cubic [§ 3 (*h*)].

(2) *Sums of Powers.* [p_0 is taken as 1.]

(*a*) $f_1(x)/f(x) = \Sigma\, 1/(x-\alpha)$

$$= \Sigma\frac{1}{x}\left(1-\frac{\alpha}{x}\right)^{-1} = \frac{n}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \dots;$$

$$\text{or} \quad = -\Sigma\frac{1}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1} = -s_{-1} - xs_{-2} - x^2 s_{-3} - \dots.$$

To obtain the first divide f_1 by f as they stand.

To obtain the second divide f_1 by f, rewriting both from R. to L.

$$(b)\ f_1(x) = \Sigma f(x)/(x-\alpha)$$
$$= \Sigma\left[x^{n-1} + (\alpha + p_1)x^{n-2} + (\alpha^2 + p_1\alpha + p_2)x^{n-3} + \dots\right]$$
$$= nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1 s_1 + np_2)x^{n-3} + \dots ;$$
$$\text{also} \quad = nx^{n-1} + (n-1)p_1 x^{n-2} + \dots .$$

Hence
$$s_1 + p_1 = 0,$$
$$s_2 + p_1 s_1 + 2p_2 = 0,$$
$$\dots\dots\dots\dots$$
$$s_m + p_1 s_{m-1} + p_2 s_{m-2} + \dots + mp_m = 0. \quad (m \not> n).$$

Note.—From these equations s_r may be written in terms of the p's only, in determinant form, *or* vice versa.

To continue the series, use

$$\Sigma\alpha f(\alpha) = 0, \quad \Sigma\alpha^2 f(\alpha) = 0, \ \&c.$$
$$s_m + p_1 s_{m-1} + p_2 s_{m-2} + \dots + p_n s_{m-n} = 0. \quad (m > n)$$

Sums of negative powers may be obtained from

$$\Sigma\frac{1}{\alpha} f(\alpha) = 0, \quad \Sigma\frac{1}{\alpha^2} f(\alpha) = 0, \ \&c.$$

$$(c)\ \log\frac{f(x)}{x^n} = \Sigma\log\left(1 - \frac{\alpha}{x}\right) = -\frac{s_1}{x} - \frac{s_2}{2x^2} - \dots .$$

Hence s's directly in terms of p's.

$$(d)\ 1 + p_1 y + p_2 y^2 + \dots = (1-\alpha y)(1-\beta y)\dots ;$$
$$\therefore\ \log(1 + p_1 y + p_2 y^2 + \dots) = -s_1 y - \tfrac{1}{2}s_2 y^2 - \dots ;$$
$$\therefore\ 1 + p_1 y + p_2 y^2 + \dots = e^{-s_1 y - \frac{1}{2}s_2 y^2 - \dots}.$$

Hence p's directly in terms of s's.

(3) *Other Integral Functions* can generally be obtained from

$$\Sigma\alpha^m\beta^n = s_m s_n - s_{m+n}; \quad \therefore\ 2\Sigma(\alpha\beta)^m = s_m^2 - s_{2m}.$$
$$\Sigma\alpha^m\beta^n\gamma^p = s_m s_n s_p - s_m s_{n+p} - s_n s_{p+m} - s_p s_{m+n} + s_{m+n+p}.$$
Hence $$6\Sigma(\alpha\beta\gamma)^m = s_m^3 - 3s_m s_{2m} + 2s_{3m}.$$

(4) *Special Devices.*

(*a*) For differences of roots of biquadratic determine cubic whose roots are $\lambda = \beta\gamma + \alpha\delta$, $\mu = \gamma\alpha + \beta\delta$, $\nu = \alpha\beta + \gamma\delta$,

i.e. $(\mu - \nu) = (\delta - \alpha)(\beta - \gamma)$, &c.

(*b*) *Equation of Difference-Squares.*

Let $(\beta-\gamma)^2$, &c., be the roots of $x^m + q_1 x^{m-1} + \ldots = 0$, $m = \frac{1}{2}n(n-1)$, and Σ_r = sum of rth powers of these.

$\Sigma (x-\alpha)^{2r} = nx^{2r} - 2rs_1 x^{2r-1} + \ldots$. Put $x = \alpha$, $x = \beta \ldots$ and add.

$\therefore\ 2\Sigma_r = ns_{2r} - 2rs_1 s_{2r-1} + \ldots$. Hence Σ's. Hence q's.

(*c*) $f(x) = (x-\alpha)(x-\beta)\ldots$ $f'(\alpha) = (\alpha-\beta)(\alpha-\gamma)\ldots$;

$\therefore\ q_m = f'(\alpha) f'(\beta) \ldots$.

12. Algebraic Solution of Cubic Equation.

$$(a_0 a_1 a_2 a_3 \mathcal{X} x\,1)^3 = 0.$$

(1) *Associated Functions.*

$$H = a_0 a_2 - a_1^2;$$

$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3;$$

$$a_0^2 \Delta = G^2 + 4H^3,$$

i.e. $\Delta = a_0^2 a_3^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3 - 3a_1^2 a_2^2$

= The 'Discriminant'.

(2) *Transformation of Equation.* $z = a_0 x + a_1$ leads to

$$z^3 + 3Hz + G = 0.$$

Notes.—(1) If α, β, γ be roots of original equation, the values of z are

$$\frac{a_0}{3}(2\alpha - \beta - \gamma), \ldots.$$

(2) The equation whose roots are $a_0^2(\beta-\gamma)^2$, $a_0^2(\gamma-\alpha)^2$, $a_0^2(\alpha-\beta)^2$ is $x^3 + 18Hx^2 + 81H^2x + 27(G^2 + 4H^3) = 0$.

(3) *Nature of Roots.* [Prove by Sturm's Theorem.]

Roots all real Δ $-^{ve}$; 'Irreducible Case'.

Roots two equal $\Delta = 0$.

Roots three equal $\Delta = 0, H = 0$; or a_0, a_1, a_2, a_3 in G.P.

(4) *Solution.* [See also § 8, xi.]

i. *Cardan-Tartaglia.* Assume $z = \sqrt[3]{p} + \sqrt[3]{q}$;

$$\therefore\ z^3 = p + q + 3\sqrt[3]{pq}\,.\,z;$$

$$\therefore\ p + q = -G,\ \sqrt[3]{pq} = -H.\ \text{Hence } p, q.$$

Notes.—(1) Apparently 6 values for z, but really only 3 different values: $\sqrt[3]{p}+\sqrt[3]{q}$, $\omega\sqrt[3]{p}+\omega^2\sqrt[3]{q}$, $\omega^2\sqrt[3]{p}+\omega\sqrt[3]{q}$.

(2) Results useless if $\Delta = -^{ve}$. Then see § 8. xi.

ii. Assume $z^3+3Hz+G = \dfrac{1}{\mu-\nu}\left[\mu(z+\nu)^3-\nu(z+\mu)^3\right]$.

iii. *Symmetric Functions.*

$$L = \alpha+\omega\beta+\omega^2\gamma,\ M = \alpha+\omega^2\beta+\omega\gamma.$$

Calculate L^3+M^3 and LM.

Roots are $\frac{1}{3}[L+M]$, $\frac{1}{3}[L\omega+M\omega^2]$, $\frac{1}{3}[L\omega^2+M\omega]$ if $\Sigma\alpha = 0$.

Notes.—(1) Available when $\Sigma\alpha \neq 0$, i.e. for untransformed equation.
(2) $L-M = (\omega-\omega^2)(\beta-\gamma)$, &c. Hence $(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$.

13. Algebraic Solution, Quartic Equation.

$$(a_0a_1a_2a_3a_4\,\mathfrak{X}\,x\,1)^4 = 0.$$

(1) *Associated Functions*

$$H = a_0a_2-a_1^2$$
$$G = a_0^2a_3-3\,a_0a_1a_2+2\,a_1^3$$
$$I = a_0a_4-4\,a_1a_3+3\,a_2^2$$
$$J = a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3$$
$$\Delta = I^3-27\,J^2$$

which are connected by relation $G^2+4H^3 = a_0^2(HI-a_0J)$.

(2) *Transformation of Equation.* $z = a_0x+a_1$ leads to

$$z^4+6Hz^2+4Gz+a_0^2I-3H^2 = 0.$$

Note.—If α, β, γ, δ be roots of original equation, roots in (2) are

$$\frac{a_0}{4}(3\alpha-\beta-\gamma-\delta),\ \&c.$$

(3) *Nature of Roots.* [Mainly proved by Sturm's Theorem.]

(i) Product of squares of differences of roots $= 256\,\Delta/a_0^6$.

(ii) $\Delta+$ Roots all real if H, $2HI-3a_0J$ both $-^{ve}$.
,, ,, imaginary if H, *or* $2HI-3a_0J$, *or* both $+^{ve}$.
$\Delta-$,, 2 real, 2 imaginary.
$\Delta = 0$ Roots 2 equal.
$I = 0$, $J = 0$ [$\because\ \Delta = 0$ also] Roots 3 equal.
$H = 0$, $I = 0$, $J = 0$ [$\because$ G and Δ also 0] Roots 4 equal.
This involves a_0, a_1, a_2, a_3, a_4 in G. P.

(4) *Solution.* All methods lead to the same 'Reducing Cubic' $4a_0^3\theta^3 - a_0 I\theta + J = 0$. If the roots of the quartic are all real this belongs to 'irreducible case'.

i. *Descartes.* Assume equation with second term removed $= (z^2 + mz + p)(z^2 - mz + q)$. Leads to cubic in m^2.

ii. *Ferrari-Simpson-Waring.* Compare $a_0(a_0a_1a_2a_3a_4)(x1)^4$ and $(a_0x^2 + 2a_1x + a_2 + 2a_0\theta)^2 - (Mx + N)^2$.

Eliminate M, N to obtain directly the reducing cubic.

iii. *Euler.* Assume $z = \sqrt{p} + \sqrt{q} + \sqrt{r}$,

$$(z^2 - p - q - r)^2 = 4(qr + rp + pq) + 8\sqrt{p}\sqrt{q}\sqrt{r}\,.z.$$

The 8 possible values of z are limited to 4 by the sign condition $2\sqrt{p}\sqrt{q}\sqrt{r} = -G$.

Note.—There are some advantages in the assumption

$$z = \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q} \quad [not\ \sqrt{qr} + \ldots]$$

which has only 4 possible values and needs no guiding sign-limitation.

From the equations $a_0\alpha + a_1 = \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p} + \sqrt{p}\sqrt{q}$

$$a_0\beta + a_1 = \sqrt{q}\sqrt{r} - \sqrt{r}\sqrt{p} - \sqrt{p}\sqrt{q}, \text{ &c.}$$

it is possible to calculate $\Pi(\alpha - \beta)^2$ and deduce $\Delta = I^3 - 27J^2$.

iv. *Other solutions.*

(*A*) *Symmetric Functions.* Form the equations whose roots are

Either (*a*) $(\beta + \gamma - \alpha - \delta)^2$, $(\gamma + \alpha - \beta - \delta)^2$, $(\alpha + \beta - \gamma - \delta)^2$,

or (*b*) $(\beta\gamma + \alpha\delta)$, $(\gamma\alpha + \beta\delta)$, $(\alpha\beta + \gamma\delta)$.

There are other possible collocations, more difficult to work with.

(*B*) *Reciprocal Equations.* Put $x = my + n$ and determine conditions that equation should be reciprocal.

14. Algebraical Solution of the Quintic and higher equations has been definitely proved to be impossible except in particular cases. The general quintic, however, may be reduced (**Tschirnhausen's Transformation**) to any one of the following forms:

$$x^5 + x + q,\ x^5 + x^2 + q,\ x^5 + x^3 + q,\ x^5 + x^4 + q,$$

or to the sum of three fifth powers.

15. THE DISCRIMINANT. $\Delta = a_0^{2n-2} \Pi (\alpha - \beta)^2$.

Methods of Calculation:

(1) $\Delta = \begin{vmatrix} s_0 & s_1 & s_2 \dots s_{n-1} \\ s_1 & s_2 \dots \dots \dots s_n \\ \vdots & \vdots & \ddots \\ s_{n-1} & s_n & s_{2n-2} \end{vmatrix}$.

(2) As the eliminant of $f(x) = 0$, $f'(x) = 0$.

(3) See § 11 (4) (*c*).

(4) For *Cubic Equations* see § 12 (1); § 12 (2), *Note* 2; § 12 (3); § 12 (4), iii, *Note* 2.

(5) For *Quartic Equations* see § 11 (4) (*a*); § 13 (1); § 13 (3); 13 (4), iii, *Note.*

16. ELIMINATION. [The methods are shown as applied to the two cubics $(p_0 p_1 p_2 p_3 \text{ℭ} x\, 1)^3 = 0$, $(q_0 q_1 q_2 q_3 \text{ℭ} x\, 1)^3 = 0$, but are perfectly general.]

Method i (**Bézout**).

$$\frac{p_0 x^3}{q_0 x^3} = \frac{p_1 x^2 + p_2 x + p_3}{q_1 x^2 + q_2 x + q_3}; \quad \frac{p_0 x^3 + p_1 x^2}{q_0 x^3 + q_1 x^2} = \frac{p_2 x + p_3}{q_2 x + q_3};$$

$$\frac{p_0 x^3 + p_1 x^2 + p_2 x}{q_0 x^3 + q_1 x^2 + q_2 x} = \frac{p_3}{q_3}.$$

Cross-multiply and eliminate x^2, x by a determinant.

Note.—If the degrees of the equations differ, bring the lower up to the higher by multiplication by a suitable power of x before applying the method.

Method ii (**Sylvester**). Multiply both equations by x^2 and also by x. From the four equations thus produced and the two original equations eliminate x^5, x^4, ... x by a determinant.

Note.—If the equations be of degree n m, the multipliers for the first are x^{m-1}, x^{m-2}, ... x, for the second x^{n-1}, x^{n-2}, ... x.

Method iii (*Symmetric Function*). Let α, β, γ be the roots of the first equation. Evaluate the symmetric function $\Pi (q_0 q_1 q_2 q_3 \text{ℭ} \alpha\, 1)^3 = 0$.

Method iv (**Euler**). It is possible to make

$(p_0 p_1 p_2 p_3 \mathfrak{X} x\,1)^3 \times (b_0 b_1 b_2 \mathfrak{X} x\,1)^2 = (q_0 q_1 q_2 q_3 \mathfrak{X} x\,1)^3 \times (a_0 a_1 a_2 \mathfrak{X} x\,1)^2$.

Multiply out, equate coefficients, and eliminate a's and b's by a determinant.

Proof.—If $x = \alpha$ be the common root

$$(p_0 p_1 p_2 p_3 \mathfrak{X} x\,1)^3 = (x-\alpha)(a_0 a_1 a_2 \mathfrak{X} x\,1)^2.$$

Note.—Leads to same determinant as *Method* ii.

Method v. If the equations are multiplied respectively by q_0, p_0 and subtracted, then by q_3, p_3 and subtracted, we obtain two quadratics, from which x may be eliminated.

Note.—This method introduces extraneous factors which do not properly belong to the eliminant. Thus in the case of the two cubics, the extraneous factor $(p_0 q_3 - p_3 q_0)$ occurs. It should never be employed unless the eliminant can be independently tested.

DETERMINANTS

[*Notation*: The symbol Δ is used for 'the determinant', or 'a determinant', or for a particular determinant. The general determinant is taken as

$$\begin{vmatrix} a_1 & b_1 & c_1 \dots & l_1 \\ a_2 & b_2 & c_2 \dots & l_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & c_n \dots & l_n \end{vmatrix}$$

and is written either $\Sigma(a_1 b_2 \dots l_n)$, the Σ applying to summation with all possible changes of suffixes, after affixing proper signs. It is also often conveniently written as

$$|\, a_1 b_2 \dots l_n \,|.]$$

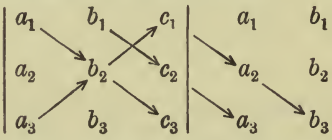
1. Rules of Formation.

i. The diagonal element (with suffixes in the proper order) is +.

ii. Any interchange of 2 suffixes (whether consecutive or not) alters the sign of the element, the letters retaining their order.

iii. Any interchange of 2 letters (whether consecutive or not) alters the sign of the element, the suffixes retaining their order.

iv. *Sarrus's Rule.* For a three-row determinant the rule indicated by the diagram is convenient.



Arrows sloping downwards indicate terms with positive signs, upwards negative.

Note.—ii and iii are equivalent statements, not distinct.

Ex. Sign of $a_3 b_7 c_6 d_5 e_1 f_4 g_2$ in $|\, a_1 b_2 c_3 d_4 e_5 f_6 g_7 \,|$. If two pairs of suffixes are interchanged, sign is unaltered. Hence sign of 3765142 is (interchanging 3 and 1, and 7 and 2) same as 1265347, i.e. as 1234657. One change only needed now. Sign minus.

2. LAW I. *If any two parallel sets of terms (not necessarily consecutive) be interchanged, the sign of* Δ *is changed.*

Proof.—From § 1, ii or § 1, iii.

Note.—The 'sets' must be complete, i.e. whole rows or whole columns.

3. LAW II. *If any two parallel sets of terms are identical or proportional,* $\Delta = 0$.

Proof.—Use Law I.

4. LAW III. Δ *is unchanged if all the rows be written as columns, in the same order, or vice versa all the columns as rows.*

Proof.—From the Rule of Formation.

5. LAW IV. *If every constituent in any row (or column) is multiplied (or divided) by the same factor,* Δ *is multiplied (or divided) by that factor.*

6. MINORS. The minor $[A_r]$ of any term a_r is the determinant formed by striking out from Δ the row and column in which that term occurs and affixing the proper sign.

Rule of Sign: Start at L. H. top corner and pass by rows or columns (never diagonally) to a_r, counting $+$, $-$ alternately at each term until a_r is reached.

Note.—Slight variations in text-books in regard to this definition. Some omit 'Rule of Sign' from definition of minor.

7. LAWS OF MINORS. (1) $A_r = \delta A_r / \delta a_r$.

(2) $\Delta = \Sigma a_r A_r$, the Σ extending
either to letters, suffixes unaltered,
or to suffixes, letters unaltered.

$= \Sigma b_r B_r = \ldots$,, ,, ,,

(3) $0 = \Sigma a_r A_s \ (r \neq s)$,, ,, ,,
$= \Sigma b_r B_s = \ldots.$

Proof.—Show that all terms of Σ occur in Δ and vice versa.

8. Higher Minors. The second minors of Δ are formed by striking out the rows and columns in which any two selected terms (not in the same row or column) occur and affixing the proper sign. Similarly for third and higher minors.

Rule of Sign. The sign is the product of the signs of the first minors of the terms selected.

9. Laws of Higher Minors.

(1) $A_{p,q} = \partial^2 \Delta / \partial a_p \, . \, \partial a_q$.

(2) $\Delta = \Sigma \mid a_p b_q \mid A_{p,q}$, the Σ extending *either* to letters, suffixes unaltered, *or* to suffixes, letters unaltered, and

$$\mid a_p b_q \mid = a_p b_q - a_q b_p,$$

being known as the *complementary* to $A_{p,q}$. [**Laplace's Theorem.**]

(3) $\Delta = \Sigma \mid a_p b_q c_r \mid A_{p,q,r}$.

(4) If in Δ which the figure represents diagrammatically A be any square block of terms, and all the terms in D (or in C, C) be zero, $\Delta = \pm A \mid BB \mid$.

C	A	C
B	D	B

10. Law V. *If every constituent in any row or column can be resolved into the sum of two others, Δ can be resolved into the sum of two others.*

Ex. If $b_1 = \beta_1 + \beta_1'$, $b_2 = \beta_2 + \beta_2' \ldots$

$$\mid a_1 b_2 c_3 \ldots l_n \mid = \mid a_1 \beta_2 c_3 \ldots l_n \mid + \mid a_1 \beta_2' c_3 \ldots l_n \mid.$$

11. Law VI. *Δ is unchanged when to the constituents of any row (or column) are added those of one or more other rows (or columns) multiplied respectively by constant factors.*

Note.—The beginner is cautioned against applying two or more of these operations simultaneously to the same rows or columns for the simplification of a numerical determinant. Thus if he adds the second row to the third row, and *simultaneously* the third row to the second, he will obtain a Δ with two rows alike, which will vanish identically. It is wise also, in any case, to leave one row or one column unaltered in each operation.

12. Law VII. *Multiplication of Determinants.*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \begin{vmatrix} \Sigma ap & \Sigma aq & \Sigma ar \\ \Sigma bp & \Sigma bq & \Sigma br \\ \Sigma cp & \Sigma cq & \Sigma cr \end{vmatrix},$$

where $\Sigma ap = a_1 p_1 + a_2 p_2 + a_3 p_3$.

Proof.—Consider the 27 determinants into which R. H. splits up.

Notes.—(1) Square of Δ = a symmetrical determinant of same order.

(2) The process is perfectly general for Δ's of any order.

(3) To multiply Δ's of different order, bring them to the same order by 'bordering' with zeros and unity.

13. Solution of Simultaneous Equations.

$$\left.\begin{array}{ll} a_1x + b_1y + c_1z + \ldots = & m_1 \\ a_2x + b_2y + \ldots = & m_2 \\ a_3x + \ldots = & m_3 \\ \ldots\ldots\ldots\ldots & \end{array}\right\},$$

where the number of equations is also the number of variables (n),

$$x = \frac{|\, m_1 b_2 c_3 \ldots l_n \,|}{|\, a_1 b_2 c_3 \ldots l_n \,|}, \quad y = \frac{|\, a_1 m_2 c_3 \ldots l_n \,|}{|\, a_1 b_2 c_3 \ldots l_n \,|}.$$

14. Eliminant of above system when

$$m_1 = m_2 = m_3 = \ldots = 0$$

is

$$|\, a_1 b_2 c_3 \ldots l_n \,| = 0.$$

15. Determinants of Minors [*Reciprocal Determinants*].

(1) $\Delta' = |\, A_1 B_2 \ldots L_n \,| = \Delta^{n-1}$.

(2) $\Delta \dfrac{\partial^2 \Delta}{\partial a_p \,.\, \partial b_q} = \dfrac{\partial \Delta}{\partial a_p} \cdot \dfrac{\partial \Delta}{\partial b_q} - \dfrac{\partial \Delta}{\partial a_q} \cdot \dfrac{\partial \Delta}{\partial b_p}$.

16. Skew Symmetrical Determinants, i.e. Δ's with leading diagonal zeros, and the terms symmetrically placed thereto equal but with opposite signs.

(1) Skew Symmetrical Determinants of an odd order vanish.

(2) Skew Symmetrical Determinants of an even order are perfect squares.

PLANE TRIGONOMETRY

1. Methods of Angular Measurement.

(1) If the same angle measures D°, G^g, θ^r,

$$D/180 = G/200 = \theta/\pi.$$

(2) The unit of *circular measure* (*radian*) is an angle at the centre of a circle which subtends on the circumference an arc equal to the radius.

Its measure is $(180/\pi)^\circ = 57^\circ$ nearly.

More exactly $57^\circ{\cdot}29578$.

(3) θ = (arc)/(radius), measured in radians.

2. Projections. The sum of the projections on any straight line of the parts of any broken line joining any two points P and Q is independent of the manner in which P and Q are joined.

Ex. Thus projection of PN + projection of NQ = projection of PQ wherever N is.

3. Ratios.

(1) $$sine = \frac{perp.}{hyp.},\ cos. = \frac{base}{hyp.},\ tan. = \frac{perp.}{base},$$

$$cosec. = \frac{hyp.}{perp.},\ sec. = \frac{hyp.}{base},\ cot. = \frac{base}{perp.}.$$

Note.—But as soon as possible the student should adopt the projection definition: If the angles be described by a straight line OP revolving from an initial position OI,

cos = ratio of projection of OP on OI, to OP,

sin = ,, ,, ,, ,, OR (at right angles to OI), to OP.

(2) $PN = OP \sin A$
$= ON \tan A,$
$ON = OP \cos A$
$= PN \cot A.$

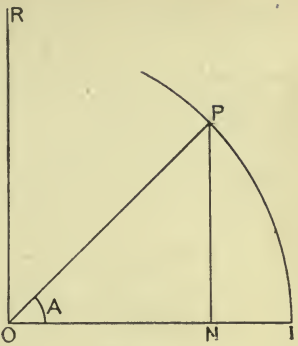


Fig. 3.

(3) cosine A = complement-sine A = $\sin(90 - A)$,
$\cot A = \tan(90° - A)$,
$\operatorname{cosec} A = \sec(90° - A)$,
$\operatorname{versin} A = 1 - \cos A$,
$\operatorname{covers} A = 1 - \sin A$.

(4) $\sin^2 A + \cos^2 A = 1,$

$$\tan A = \frac{\sin A}{\cos A},$$
$$\sec^2 A = 1 + \tan^2 A,$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

(5) If $\tan A = \frac{a}{b}$, $\sin A = \frac{a}{\sqrt{a^2+b^2}}$, $\cos A = \frac{b}{\sqrt{a^2+b^2}}$.

Note.—The ambiguity of the root must be remembered.

(6) The student should thoroughly practise himself in the formulae, giving all the ratios in terms of one selected ratio. These are recalled mentally as follows: To write all the ratios in terms of the cosecant think of a right-angled Δ with hypotenuse c and perpendicular 1, then the base is $\sqrt{c^2-1}$, and we have the results

$$\sin A = \frac{1}{c}, \cos A = \frac{\sqrt{c^2-1}}{c}, \tan A = \frac{1}{\sqrt{c^2-1}}, \text{ \&c.}$$

Here c stands for *cosecant*. The ambiguity of the root must always be remembered.

4. Values of Special Ratios.

(1)	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Note.—The first line may be remembered by the mnemonic

$$\sqrt{\tfrac{0}{4}}, \sqrt{\tfrac{1}{4}}, \sqrt{\tfrac{2}{4}}, \sqrt{\tfrac{3}{4}}, \sqrt{\tfrac{4}{4}}.$$

The second line is the first written in reverse order. The third is the first divided by the second. But as soon as possible these results must be individually known.

(2) $\quad \text{Sin } 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}, \qquad \sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4},$

$$\sin 18° = \frac{\sqrt{5}-1}{4}, \qquad \sin 54° = \frac{\sqrt{5}+1}{4},$$

$$\tan 15° = 2-\sqrt{3}, \qquad \tan 75° = 2+\sqrt{3},$$

$$\tan 22\tfrac{1}{2}° = \sqrt{2}-1, \qquad \tan 67\tfrac{1}{2}° = \sqrt{2}+1.$$

Note.—Since $3° = 18° - 15°$ the ratios of all multiples of 3° can be explicitly expressed in quadratic surds.

5. Angles of Form $n\frac{\pi}{2} \pm A$.

(1) Any trigonometrical ratio of $n\frac{\pi}{2} \pm A$ will be the $\begin{matrix}\text{same}\\ \text{complementary}\end{matrix}$ ratio of A according as n is $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$. To affix the sign, suppose A acute, the sign will be that of the *original* ratio in the quadrant in which the compound angle lies.

Notes.—(1) Too much importance cannot be attached to this formula. It must be practised until the results can be written without hesitation. The sign determined by supposing A acute holds when it is not.

(2) In applying it to determine such expressions as sin 225°, &c., refer to the nearest *even* multiple of $\pi/2$, e.g. treat this angle as $180° + 45°$ and not as $270° - 45°$.

(3) There are two distinct types of proof, according as n is even or odd.

(2) *Simplest Cases*:

$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A,$$
$$\sin(90°-A) = \cos A, \quad \cos(90°-A) = \sin A,$$
$$\sin(180°-A) = \sin A, \quad \cos(180°-A) = -\cos A.$$

6. Angles with the same ratios as A.

(1) If $\sin\theta = \sin A$, $\theta = n\pi + (-)^n A$. So for cosecants.

(2) If $\cos\theta = \cos A$, $\theta = 2n\pi \pm A$. So for secants.

(3) If $\tan\theta = \tan A$, $\theta = n\pi + A$. So for cotangents.

Note.—*Always* use these values in writing the solution of trigonometrical equations, avoiding, where possible, the sine formula in favour of either the cosine or the tangent formula.

7. Addition Formulae.

$$\begin{aligned}
(1)\quad \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B,\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B,\\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.\\
\tan(45° \pm A) &= \frac{1 \pm \tan A}{1 \mp \tan A}.
\end{aligned}$$

Notes.—(1) The cotangent formula should *not* be learnt. It is

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}.$$

(2) The proof should be the Projection Proof.

(3) The sine and cosine results follow easily also from Ptolemy's Theorem (*GEOM.* § 24 (4)).

$$\begin{aligned}
(2)\quad \sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},\\
\sin A - \sin B &= 2\sin\frac{A-B}{2}\cos\frac{A+B}{2},\\
\cos A + \cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},\\
\cos A - \cos B &= -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}.
\end{aligned}$$

These results *must* be remembered in words:

Sum of two sines = 2 *sin* (*half sum*) *cos* (*half diff.*),
Diff. of two sines = 2 *sin* (*half diff.*) *cos* (*half sum*),
Sum of two cosines = 2 *cos* (*half sum*) *cos* (*half diff.*),
Diff. of two cosines = MINUS 2 *sin* (*half diff.*) *sin* (*half sum*).

Notes (1) The student must be able to write mentally and without intermediate steps such results as $\tan A + \tan B = \sin(A+B)/\cos A \cos B$,

$\tan A + \cot 3A = \cos 2A / \cos A \sin 3A$, &c. They are not to be learnt as formulae.

(2) The following transformation is often valuable:

$$\sin A \pm \cos A = \sqrt{2} \sin (A \pm 45°) = \sqrt{2} \cos (A \mp 45°).$$

(3) $$\sin A \sin B = \tfrac{1}{2}\{\cos (A - B) - \cos (A + B)\},$$
$$\cos A \cos B = \tfrac{1}{2}\{\cos (A - B) + \cos (A + B)\},$$
$$\sin A \cos B = \tfrac{1}{2}\{\sin (A - B) + \sin (A + B)\}.$$

These results *must* be remembered in words:

Product of two sines $= \tfrac{1}{2}\{\cos \text{ diff.} - \cos \text{ sum}\}$,
Product of two cosines $= \tfrac{1}{2}\{\cos \text{ diff.} + \cos \text{ sum}\}$,
Product of sin and cos $= \tfrac{1}{2}\{\sin \text{ sum} + \sin \text{ diff.}\}$.

Note.—In the phrase 'difference' the angles are to be taken in order. Thus if in the last formula the *sine* has the smaller angle, the difference and its sine will be negative.

(4) In a trigonometrical simplification involving products of sines or cosines the proper procedure is *almost always* to use the formulae of (3).

(5) $$\sin (A + B + C + \ldots) = \Sigma_1 - \Sigma_3 + \Sigma_5 - \ldots,$$
$$\cos (A + B + C + \ldots) = \Sigma_0 - \Sigma_2 + \Sigma_4 - \ldots,$$

where $\Sigma_0 = \cos A \,.\, \cos B \,.\, \cos C \ldots$ and $\Sigma_r =$ the sum of all possible terms obtained by changing r of the cosines of Σ_0 into sines.

$$\tan (A + B + C + \ldots) = \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots},$$

where $S_r =$ the sum of the products of $\tan A$, $\tan B$, $\tan C, \ldots$ taken r together.

8. Multiple Angles.

(1) $$\sin 2A = 2 \sin A \cos A = \frac{2t}{1 + t^2}, \qquad [t = \tan A]$$

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{1 - t^2}{1 + t^2},$$
$$= 2\cos^2 A - 1 \,;\; 1 + \cos 2A = 2\cos^2 A,$$
$$= 1 - 2\sin^2 A \,;\; 1 - \cos 2A = 2\sin^2 A,$$
$$\tan 2A = \frac{2t}{1 - t^2}; \quad \tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

(2) $\sin^2 A - \sin^2 B = \sin(A-B)\sin(A+B)$.

(3) $\sin 3A = 3s - 4s^3$, $[s = \sin A]$

$\cos 3A = 4c^3 - 3c$, $[c = \cos A]$

$\tan 3A$: See § 7 (5). Put $A = B = C$.

For higher multiples see § 24.

(4) In a trigonometrical simplification involving squares or higher powers of sines and cosines the proper procedure is *almost always* to transform the powers into multiple angles. [§ 23 (1).]

(5) $$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A},$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}.$$

To remove the ambiguity see § 7 (2), *Note* (2).

$$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} = \sqrt{\frac{1-\cos A}{1+\cos A}}.$$

9. Inverse Functions.

(1) $\mathrm{Sin}^{-1}x + \cos^{-1}x = \frac{\pi}{2}$. So for all complementary functions.

Note.—It is understood here that $\sin^{-1}x$ means the *least* value of the angle. But the symbol is also used for any value, so that

$$\sin^{-1}\tfrac{1}{2} = n\pi + (-)^n\frac{\pi}{6}.$$

(2) $$\tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab}.$$

Note.—The student must practise the reading of other formulae as inverses, e.g. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$.

10. Properties of a Triangle.

(1) $A + B + C = \pi$. Hence $\sin A = \sin(B+C)$, &c.

(2) $\Sigma\tan A = \tan A\tan B\tan C$,

$$\Sigma\cot\frac{A}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$

$$\Sigma\cos^2 A = 1 - 2\cos A\cos B\cos C.$$

Notes.—(1) The following are for reference only :

$$\Sigma \sin A = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$$

$$\Sigma \cos A = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$

$$\Sigma \sin 2A = 4 \sin A \sin B \sin C,$$

$$\Sigma \cos 2A = -1 - 4 \cos A \cos B \cos C.$$

With a little practice they can easily be obtained mentally, as required.

(2) Since $(\pi - 2A) + (\pi - 2B) + (\pi - 2C) = \pi$, the values $\pi - 2A$, $\pi - 2B$, $\pi - 2C$ can be substituted for A, B and C. So also $(2\pi - 5A)$, &c.

(3) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d.$$

Notes.—(1) The substitutions $a = d \sin A$, $b = d \sin B$, $c = d \sin C$ are the most valuable of all in triangle-identities.

(2) d is (see § 12 (1)) the diameter of the circumcircle $= 2R$.

(4) $a = b \cos C + c \cos B$, &c.

(5) $a^2 = b^2 + c^2 - 2bc \cos A$, &c.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ &c.}$$

(6) $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ &c. ;}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

(7) $$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}, \text{ &c.}$$

(8) *Area of* $\Delta = \Delta = \frac{1}{2} bc \sin A =$ &c.

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

11. Solution of Triangles.

(1) Given a, b, c. Use § 10 (6) or § 10 (5).

(2) Given A, b, c. Use § 10 (7) to find $B - C$; $B + C = 180° - A$; or § 10 (5) for a.

(3) Given A, B, c. $C = 180° - A - B$, then use § 10 (3).

(4) Given A, B, a. As in (3).

(5) Given a, b, A we have the '*Ambiguous Case*', of which there are three treatments possible, all of which should be familiar.

(*a*) *Geometrical.* The circle, centre C, radius a, either does not meet the known direction of c—no solution;

or touches it—one solution, $B = 90°$;

or cuts it in two points on same side of A—two solutions, and two values of B, i.e. B and $180 - B$;

or cuts it in two points on opposite sides of A—one solution.

(*b*) *Use* $\sin B = \frac{b}{a} \sin A$. Data make $\frac{b}{a} \sin A > 1$ (no Δ), $= 1$ (one Δ), < 1 (two Δ's or one as $b >$ or $\not> a$).

(*c*) *Use* $a^2 = b^2 + c^2 - 2bc \cos A$ to give quadratic for c.

12. Related Circles.

(1) *Circumcircle.* $R = \dfrac{a}{2 \sin A} = \ldots = \dfrac{abc}{4\Delta}$.

(2) *Incircle.* $r = \dfrac{\Delta}{s} = \dfrac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \&c.$

$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = (s - a) \tan \frac{A}{2}.$$

(3) *A-excircle.* $r_1 = \dfrac{\Delta}{s - a} = \dfrac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$

$$= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = s \tan \frac{A}{2}.$$

Note.—The formulae may be derived from the incircle by regarding the excircle as the incircle of a 'triangle' whose angles are

$$A, \pi - B, \pi - C,$$

these being the angles bisected to obtain the centre.

(4) *Distance of Circumcentre and Incentre* $= \sqrt{R^2 - 2Rr}$.
Distance of Circumcentre and Excentre $= \sqrt{R^2 + 2Rr_1}$.

Note.—Hence $r < \frac{1}{2}R$.

(5) *Contacts of Incircle* X, Y, Z. $BX = s - b$, $CX = s - c$.
Contacts of A-excircle with AB, AC, are distant s from A.

13. Related Triangles and Lines.

(1) *The Pedal Triangle.* [See *GEOM.* § 16.]

The sides of the pedal triangle are $a\cos A$, $b\cos B$, $c\cos C$.
The angles „ „ „ $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.
Its incentre is the orthocentre of the original triangle.
Its excentres are the angular points of the original triangle.
Its circumcircle is the N. P. C. of the original triangle.

Note.—Hence any property of a Δ remains true if we substitute $a\cos A$,... for a, b, c and $\pi - 2A$,... for A, B, C.

(2) *Nine Point Circle.* [See *GEOM.* § 17.]

If O be circumcentre, G the centroid, N the N. P. centre, and H the orthocentre, O, G, N, H are collinear, $OG = \frac{1}{3}OH$, $ON = \frac{1}{2}OH$.

The Nine Points are the three feet of the perpendiculars, three midpoints of sides, and the three midpoints of lines joining H to vertices.

The radius is $\frac{1}{2}R$.

G and H are centres of similitude of N. P. C. and circumcircle.

The N. P. C. touches the in- and all the ex-circles (**Feuerbach's Theorem**). [See *GEOM.* § 17 (5).]

Note. The following distances are given for reference:

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C),$$
$$IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C.$$

From these and $OI^2 = R^2 - 2Rr$ it is easy to prove $NI = \frac{1}{2}R - r$ and thus deduce Feuerbach's Theorem.

(3) *The Medians.* [See *GEOM.* § 12.]

Squares of lengths $\frac{1}{4}(2b^2 + 2c^2 - a^2)$, ...

Inclinations to sides $\cot\theta = \frac{1}{2}(\cot B - \cot C)$ ....

(4) *The Angle Bisectors.* [See *GEOM.* § 15.]

The internal bisector of A divides BC into segments $ac/(b+c)$, $ab/(b+c)$.

The external bisector of A divides BC into segments $ac/(b-c)$, $ab/(b-c)$.

Their lengths are $2bc\cos\frac{A}{2}/(b+c)$ and $2bc\sin\frac{A}{2}/(b-c)$ respectively.

14. QUADRILATERALS. $(A + C = 2\alpha.)$

(1) Area $= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$.

(2) If the quadrilateral is cyclic $A + C = B + D = \pi$.

If the quadrilateral is circumcyclic $a + c = b + d$.

Other results for reference:

If e, f are the diagonals and ϕ the angle between them,

(3) $e^2f^2 = a^2c^2 + b^2d^2 - 2abcd\cos 2\alpha$.

(4) $\Delta = \frac{1}{2}ef\sin\phi$,

$= \frac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan\phi$,

$= \frac{1}{4}\sqrt{4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2}$.

(5) For a cyclic quadrilateral the sqs. on the diagonals are

$(ac+bd)(ad+bc)/(ab+cd)$, $(ab+cd)(ac+bd)/(ad+bc)$

and the circumradius is $\frac{1}{4\Delta}\sqrt{(ab+cd)(ac+bd)(ad+bc)}$.

15. REGULAR POLYGONS. The following results should be obtained mentally when required:

(1) $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$.

(2) $r = \frac{a}{2}\cot\frac{\pi}{n}$.

(3) $\Delta = \frac{n}{2}R^2\sin\frac{2\pi}{n} = \frac{n}{4}a^2\cot\frac{\pi}{n}$.

16. MAXIMA AND MINIMA.

(1) $a\sin x + b\cos x = \sqrt{a^2 + b^2}\sin(x + \alpha)$, where $\cot\alpha = a/b$.

Hence, &c.

(2) Given $\Sigma A = \alpha$, to investigate the max. or min. values of:

$\Sigma \sin A$. $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$, $\therefore$ increased without altering $A+B$ by making $A = B$. Max. $A = B = C \ldots$

$\Pi \sin A$. $2 \sin (A+B) = \cos (A-B) - \cos (A+B)$.

Max. $A = B = C, \ldots$

$\Sigma \tan A$. $\tan A + \tan B = \dfrac{2 \sin (A+B)}{\cos (A-B) + \cos (A+B)}$. Min. $A = B = C$.

$\Pi \tan A$. $\tan A \tan B$

$= \dfrac{\cos (A-B) - \cos (A+B)}{\cos (A-B) + \cos (A+B)} = 1 - \dfrac{2 \cos (A+B)}{\cos (A-B) + \cos (A+B)}$, &c.

17. Limiting Values.

(1) If $\theta < \frac{1}{2}\pi$, $\sin \theta$, θ, $\tan \theta$ are in ascending order.

(2) $\mathrm{Lt}_{\theta=0} \left(\dfrac{\sin \theta}{\theta}\right) = 1$, $\mathrm{Lt}_{\theta=0} \left(\dfrac{\tan \theta}{\theta}\right) = 1$.

(3) $\theta > \sin \theta > \theta - \frac{1}{6}\theta^3$,

$1 > \cos \theta > 1 - \frac{1}{2}\theta^2$.

(4) As θ increases from 0 to $\frac{1}{2}\pi$, $\sin \theta/\theta$ continually diminishes and $\tan \theta/\theta$ continually increases.

(5) *Euler's Theorem*:

$$\cos \frac{\theta}{2} \cdot \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \ldots \text{ to } \infty = \frac{\sin \theta}{\theta}.$$

18. Series. [See also § 21 (7).]

(1) $\begin{matrix}\sin \\ \cos\end{matrix} \alpha + \begin{matrix}\sin \\ \cos\end{matrix} (\alpha+\beta) + \ldots$ to n terms.

$$\text{Sum} = \frac{\sin (\text{number} \times \frac{1}{2} \text{diff.})}{\sin (\frac{1}{2} \text{diff.})} \begin{matrix}\sin \\ \cos\end{matrix} (\text{mean of first and last}).$$

(2) $\Sigma \sin^p (\alpha + n\beta) \cos^q (\alpha + n\beta)$. Convert into multiple angles. [See § 8 (4) and for method § 23 (1).]

e.g. $\Sigma \sin^2 \alpha$. Use $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$.

(3) $\sin \alpha - \sin (\alpha+\beta) + \sin (\alpha+2\beta) \ldots$, is treated as

$$\Sigma \sin (\alpha + n\beta + n\pi).$$

(4) $\underset{\cos}{\sin}\alpha + c\underset{\cos}{\sin}(\alpha+\beta) + c^2\underset{\cos}{\sin}(\alpha+2\beta) + \ldots$. The sum to infinity beginning at any term

$$= \frac{\text{First Term} - c^2 \times \text{'Term before first'}}{1 - 2c\cos\text{diff.} + c^2}.$$

Notes.—(1) 'Term before first' in given series would be $\frac{1}{c}\sin(\alpha-\beta)$, &c.

(2) Deduce sum to n terms, namely $\sum_1^\infty - \sum_{n+1}^\infty$.

Proof.—It is a recurring series whose scale of relation is

$$1 - 2c\cos\beta + c^2.$$

(5) $u_0\cos\alpha + u_1\cos(\alpha+\beta) + u_2\cos(\alpha+2\beta) + \ldots$, where u_r is a rational integral function of r. To sum, multiply by $\cos\beta$ and subtract; the result will be a series of one lower degree in r in the coefficients. Similarly for sine series.

e.g. $\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \ldots$.

(6) (*a*) $\operatorname{cosec} x + \operatorname{cosec} 2x + \ldots$. Use $\operatorname{cosec} 2x = \cot x - \cot 2x$.

(*b*) $\tan x + \frac{1}{2}\tan\frac{x}{2} + \ldots$. Use $\tan x = \cot x - 2\cot 2x$.

(*c*) $\tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \ldots$

Use $\tan^{-1}(x+1) - \tan^{-1}(x-1) = \tan^{-1}\frac{2}{x^2}$.

(*d*) $\tan x\tan 2x + \tan 2x\tan 3x + \ldots$.

Use $\tan A\tan B = \frac{\tan A - \tan B}{\tan(A-B)} - 1$, &c.

19. De Moivre's Theorem.

(1) $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

and so for any number of factors.

$$(\cos\alpha + i\sin\alpha)/(\cos\beta + i\sin\beta) = \cos(\alpha-\beta) + i\sin(\alpha-\beta).$$

(2) $\cos\theta + i\sin\theta$ is one of the values of

$$(\cos n\theta + i\sin n\theta)^{1/n}$$

and the remaining $n-1$ values are obtained by writing $2r\pi + n\vartheta$ for $n\theta$ and $r = 1, 2, 3 \ldots (n-1)$.

(3) *cos nθ and sin nθ expanded homogeneously* in cos θ, sin θ.

Method : In the identity $\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$ expand and equate real and imaginary parts on the two sides.

Results :
$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta \sin^4\theta \ldots,$$
$$\sin n\theta = n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta \sin^3\theta + \ldots,$$
$$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{3!}\tan^3\theta + \ldots}{1 - \frac{n(n-1)}{2!}\tan^2\theta + \ldots}.$$

Note.—These results hold for all integral values of n, and for *all* values of n provided θ lies between $\pm\frac{1}{4}\pi$. When n is integral they can be deduced from § 7 (5). The series terminate in this case only.

(4) *Trigonometrical Identities.* Any algebraical identity in a, b, c, &c., gives rise to two trigonometrical identities by substitution, $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$, &c.

Ex. $\Sigma a^2(b-c) = -(b-c)(c-a)(a-b)$ gives
$$\Sigma \cos(\beta+\gamma+2\alpha)\sin(\beta-\gamma) = 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\cos(2\alpha+2\beta+2\gamma),$$
$$\Sigma \sin(\beta+\gamma+2\alpha)\sin(\beta-\gamma) = 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin(2\alpha+2\beta+2\gamma).$$

20. Expansions of sin θ, cos θ, &c.

(1)
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots,$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots.$$

Notes.—(1) These results are true for *all* values of θ, provided θ be in radians.

(2) In the proof it is necessary to prove that Lt $(\sin\theta/\theta)^n = 1$, and also that Lt $(\cos\theta)^n = 1$ when $n = \infty$ and $\theta = 0$. Another point of difficulty is generally omitted in elementary treatment. [*Hobson*, § 99.]

(3) To expand $\sin^2\theta$, $\sin^3\theta$, $\sin^2\theta\cos\theta$, &c., use Multiple Angles. [§ 8 (4).]

$$(2)\ \tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \ldots.$$

$$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \frac{2\theta^5}{945} - \ldots.$$

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \frac{\theta}{6} + \frac{7\theta^3}{360} + \frac{31\theta^5}{15120} + \ldots.$$

$$\sec\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \frac{61\theta^6}{720} + \ldots. \quad [\text{See } \textit{DIFF. CALC.} \S\ 7\ (6)\ (b).]$$

21. Exponential Values.

$$(1)\ \cos\theta + i\sin\theta = e^{\theta i},$$
$$\cos\theta - i\sin\theta = e^{-\theta i}.$$

$$(2)\ \cos\theta = \frac{1}{2}(e^{\theta i} + e^{-\theta i}), \quad \sin\theta = \frac{1}{2i}(e^{\theta i} - e^{-\theta i}).$$

(3) Hence exponential values for all the ratios.

(4) *Complex Quantities.* The following is a summary of method: If x is complex *define* e^x as $1 + x + x^2/2! + x^3/3! + \ldots$ after proving the series convergent. Prove $e^x \cdot e^y = e^{x+y}$. *Define* $\sin x$, $\cos x$ by the series values. Deduce results of § 21 (2) for complex values. From these deduce addition theorems for $\sin(x+y)$ and all formulae founded thereon, thence periodicity of $\sin x$ and $\cos x$. Tan x is defined as $\sin x/\cos x$. For logs *define* x as a log of a where $a = e^x$. Hence since $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi$, $x + 2n\pi i = \log a$. *Define* a^x as $e^{x \log a}$. Deduce ordinary series for $\log(1+y)$ when y is complex, the series holding only under certain convergency conditions.

(5) *Gregory's Series.* $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots.$
$[1 > x > -1.]$

(6) *Evaluation of* π. Apply Gregory to the following:

Euler. $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

Machin. $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

Clausen. $\frac{1}{4}\pi = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$.

Rutherford. $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.

(7) *Summation of Series.* [See also § 18.]

If the sum of the series $a_0 + a_1 x + a_2 x^2 + \ldots$ is known, $f(x)$ say, that of the series

$$a_0\cos\alpha + a_1 x\cos(\alpha+\theta) + a_2 x^2\cos(\alpha+2\theta) + \ldots$$

can be determined. Similarly for the sine series.

Note.—The cosine series is the real part of $e^{\alpha i} f(xe^{\theta i})$.

(8) *Expansions.* The exponential values are applied to such questions as:

(*a*) Expand $e^{ax}\sin(bx+c)$ in ascending powers of x.

(*b*) Given $\sin x = k\sin(x+\alpha)$ expand x in powers of k (<1).

(*c*) If $(1-k)\tan x = (1+k)\tan y$ ($k<1$) expand x in terms of y or y in terms of x.

22. Hyperbolic Functions.

(1) *Definitions.* $\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$, $\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$, $\tanh\theta = \sinh\theta/\cosh\theta$, &c.

(2) $\cosh^2\theta - \sinh^2\theta = 1$.

(3) $\cosh\theta = \cos\theta i$, $\sinh\theta = -i\sin\theta i$. Hence all results in ordinary trigonometry may be turned into hyperbolic results. Some of these are added for reference below.

(4) The hyperbolic sines and cosines are periodic with period $2\pi i$. The hyperbolic tangent is periodic with period πi.

(5) $\cosh^{-1}x = \log(x+\sqrt{x^2-1})$.
$\sinh^{-1}x = \log(x+\sqrt{x^2+1})$.
$\tanh^{-1}x = \frac{1}{2}\log[(1+x)/(1-x)]$.

(6) *For Reference*:
$\cosh 0 = 1$; $\sinh 0 = 0$.
$\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x\sinh y$.
$\cosh(x\pm y) = \cosh x\cosh y \pm \sinh x\sinh y$.
$\tanh(x+y+\ldots) = \dfrac{S_1+S_3+S_5+\ldots}{1+S_2+S_4+\ldots}$.
$\sinh 2x = 2\sinh x\cosh x$.
$\cosh 2x = \cosh^2 x+\sinh^2 x$.
$\sinh 3x = 3\sinh x+4\sinh^3 x$.
$\cosh 3x = 4\cosh^3 x-3\cosh x$.

All the above and many more may be derived from the ordinary form by the rule that if *sinh* occurs explicitly or implicitly (as in tanh x) *twice or thrice*, it occurs with a sign opposite to that in the corresponding formula in Plane Trigonometry.

Notes.—(1) Some curious logarithmic series arise from the expansions of $\sinh^{-1}x$ and $(\sinh^{-1}x)^2$ [see § 25 (1) (2)] by the use of § 22 (5).

(2) Just as, if $OA = a$ be the radius of a circle, centre O, and PN, ON be ordinate and abscissa of a point P, $\angle AOP = \theta$, $ON = a\cos\theta$, $PN = a\sin\theta$, area $AOP = \frac{1}{2}a^2\theta$, so if OA be the semi-major axis, length a, of a R.H. centre O, and P be a point on it with ordinate and abscissa PN, ON, $ON = a\cosh\phi$, $PN = a\sinh\phi$, area $AOP = \frac{1}{2}a^2\phi$, but ϕ is *not* the angle AOP. Hence the name *hyperbolic*.

23. EXPANSION OF $\sin^n\theta$, $\cos^n\theta$ IN MULTIPLE ANGLES.

(1) $\sin^n\theta$, $\cos^n\theta$, $\sin^m\theta\cos^n\theta$ and such expressions are expanded by the use of the results that if $x = \cos\theta + i\sin\theta$,

$$x + 1/x = 2\cos\theta, \qquad x^n + 1/x^n = 2\cos n\theta,$$
$$x - 1/x = 2i\sin\theta, \qquad x^n - 1/x^n = 2i\sin n\theta.$$

Care is needed in the last terms, which are sometimes without the factor 2 which occurs in the other terms.

Ex. $\sin^3\theta\cos^2\theta = \dfrac{1}{(2i)^3}\cdot\dfrac{1}{2^2}\left(x-\dfrac{1}{x}\right)^3\left(x+\dfrac{1}{x}\right)^2$

$= -\dfrac{1}{32i}\left[x^5 - \dfrac{1}{x^5} - \left(x^3 - \dfrac{1}{x^3}\right) - 2\left(x - \dfrac{1}{x}\right)\right]$

$= -\frac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta).$

1 −3 +3 −1
1 −2 +0 +2 −1
1 −1 −2 +2 +1 −1

(2) *Results* (for reference):

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots.$$

$$2^{n-1}\sin^n\theta\cos\frac{n\pi}{2} = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta - \ldots$$

[n even]

$$2^{n-1}\sin^n\theta\sin\frac{n\pi}{2} = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{2!}\sin(n-4)\theta - \ldots$$

[n odd].

Notes.—(1) The above results are true when n is integral, provided that *the last term* as given by the regular sequence is *halved* if it is not accompanied by $\sin\theta$ or $\cos\theta$.

(2) They are also true if n is not an integer *if the L.H. is doubled,* i. e. 2^n written for 2^{n-1} and if θ lies between $\pi/2$ and 0.

(3) The complete investigation when n is not integral is difficult. [*Hobson*, § 222.] A rough investigation may start for first series from R.H. = real part of $x^n + nx^{n-2} + \dfrac{n(n-1)}{2!}x^{n-4} + \ldots$, i. e. of $(x + 1/x)^n$, &c.

24. EXPANSION OF $\sin n\theta$, $\cos n\theta$ IN POWERS OF s AND c.

A. Descending Powers.

Methods. (i) $(1 - zx)(1 - z/x) = 1 - z(2\cos\theta - z)$

if $\quad x = \cos\theta + i\sin\theta.$

Take logs and equate coefficient z^n for $\cos n\theta$ in powers of $\cos\theta$. Deduce $\sin n\theta$ by writing $\frac{1}{2}\pi - \theta$ for θ *or* directly from $(1 - zx)(1 + z/x) = 1 - z(z + 2i\sin\theta)$.

(ii) Sum the series $x \begin{matrix}\sin\\ \cos\end{matrix} \theta + x^2 \begin{matrix}\sin\\ \cos\end{matrix} 2\theta + \ldots$ [§ 18 (4)], and equate coefficients of x^n in the expanded sum.

(iii) From § 19 (3) by $\sin^2\theta = 1 - \cos^2\theta$. [*Hobson*, § 78.]

Results:

(1) $\sin n\theta / \sin\theta = (2c)^{n-1} - (n-2)(2c)^{n-3} + \dfrac{(n-3)(n-4)}{2!}(2c)^{n-5} - \ldots$.

(2) $2\cos n\theta$

$= (2c)^n - n(2c)^{n-2} + \dfrac{n(n-3)}{2!}(2c)^{n-4} - \dfrac{n(n-4)(n-5)}{3!}(2c)^{n-6} + \ldots$.

(3) Expansions in terms of $\sin\theta$ by putting $\frac{1}{2}\pi - \theta$ for θ. They differ as n is even or odd.

Note.—The expansions are to cease just before any index becomes negative.

B. Ascending Powers.

Methods. A. (i) and (ii) are available, or the results of *A.* may be rearranged.

(iii) Assume $\cos n\theta = a_0 + a_2 s^2 + a_4 s^4 + \ldots$. Differentiate twice and equate coefficients of s^{2r} to obtain a relation between the coefficients. Similarly for other results.

Note.—This is on the whole the easiest way. For Differentiation see *ALG.* § 30 (9) *Note.*

Results:

n even: (1) $\cos n\theta = 1 - \dfrac{n^2}{2!}s^2 + \dfrac{n^2(n^2-2^2)}{4!}s^4 - \ldots$;

(2) $\sin n\theta = nc\left[s - \dfrac{n^2-2^2}{3!}s^3 + \dfrac{(n^2-2^2)(n^2-4^2)}{5!}s^5 - \ldots\right]$,

and results for $\cos n\theta$, $\sin n\theta / \sin\theta$ in terms of c by writing $\frac{1}{2}\pi - \theta$ for θ.

n odd: (3) $\cos n\theta = c\left[1 - \dfrac{n^2-1^2}{2!}s^2 + \dfrac{(n^2-1^2)(n^2-3^2)}{4!}s^4 - \ldots\right]$;

(4) $\sin n\theta = ns\left[1 - \dfrac{n^2-1^2}{3!}s^2 + \dfrac{(n^2-1^2)(n^2-3^2)}{5!}s^4 - \ldots\right]$,

and results for $\sin n\theta / \sin\theta$, $\cos n\theta$ in terms of c by writing $\frac{1}{2}\pi - \theta$ for θ.

Notes.—(1) The results given, *B* (1), (2), (3), (4), hold for any value of θ between $\pm\frac{1}{2}\pi$. Also for all values of n, but terminate only when n is even or odd integer respectively.

(2) Note results when $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

(3) Many interesting deductions. Multiply *B.* (4) by i and add to *B.* (1). Put $i \sin\theta = x$. Hence expansion in x of $[x+\sqrt{(1+x^2)}]^n$. This can be proved independently by double differentiation as in *B.* (iii). In this result put $x = \cot 2\phi$. Hence expansion of $\cot^n\phi$ in powers of $\cot 2\phi$. Similar results from *B.* (2) and *B.* (3).

25. Expansion of $(\sin^{-1}x)$.

$$(1)\quad \sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \ldots. \qquad [x \not> 1$$

Easily remembered from $(1-x^2)^- = 1 + \frac{1}{2}.x^2 + \frac{1.3}{2.4}x^4 + \ldots$.

$$(2)\quad \tfrac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2.4}{3.5}\cdot\frac{x^6}{6} + \ldots. \qquad [x \not> 1$$

Easily remembered if (1) be written $\dfrac{(\sin^{-1}x)^1}{1} = \dfrac{x^1}{1} + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \ldots$ by increasing every *number* by unity, turning 1 into 2, 2 into 3, &c.

Proof.—Expand $\sin n\theta = n\theta - \frac{1}{6}n^3\theta^3 + \ldots$ in § 24 *B.* (4) and equate coefficients of n. For (2) take § 24 *B.* (1) and equate coefficients of n^2.

Notes.—(1) $\sin^{-1}x/\sqrt{1-x^2}$ is derived from § 24 *B.* (2) and $(\sin^{-1}x)^2/\sqrt{1-x^2}$ from § 24 *B.* (3).

(2) Note series for π, π^2, &c., which arise from the formulae of § 25, by substitutions $x = 1$, $\frac{1}{2}$, $1/\sqrt{2}$, &c.

26. Symmetrical Functions.

(1) S. F. of $\cos\pi/n$, $\cos 3\pi/n$, $\cos 5\pi/n, \ldots \cos(2n-1)\pi/n$ are roots in c, $= \cos\theta$, of $\cos n\theta = -1$ [§ 24 *A.* (2)]. These will pair off into repeated roots with a possible middle term $c = -1$. Hence S. F. such as

$$\Sigma\cos\pi/n,\ \Sigma\sec\pi/n,\ \Sigma\sec^2\pi/n,\ \Sigma\tan^2\pi/n.$$

(2) If the equation giving $\cos n\theta$ in powers of c is regarded as equation in c its roots are

$$\cos\theta,\ \cos(\theta + 2\pi/n),\ \cos(\theta + 4\pi/n)\ldots.$$

Hence S. F. of these.

(3) So for $\sin\theta$, $\sin(\theta + 2\pi/n)$, $\sin(\theta + 4\pi/n)\ldots$.

(4) The roots in t of

$$\tan n\theta\left[1-\frac{n(n-1)}{2!}t^2+\dots\right]=nt-\frac{n(n-1)(n-2)}{3!}t^3+\dots$$

are $\tan\theta$, $\tan(\theta+\pi/n)$, $\tan(\theta+2\pi/n)$.... Hence S. F. of these and of $\sec^2\theta$, $\sec^2(\theta+\pi/n)\dots$, and of

$$\cot\theta,\ \cot(\theta+\pi/n)\dots.$$

(5) For functions of *a portion only* of the complete set of angles, e.g. $\phi=\pi/13$, $x=\cos\phi+\cos3\phi+\cos9\phi$, $y=\cos5\phi+\cos7\phi+\cos11\phi$. Prove [§ 7 (4), § 8 (4)] $2x^2=3-2y-x$, $2y^2=3-y-2x$ and deduce $x,\ y=\frac{1}{4}(1\pm\sqrt{13})$. For another illustration see *TH. EQNS.* § 9 (8).

27. Factors.

(1) $\cos n\theta=2^{n-1}(\cos\theta-\cos\pi/2n)(\cos\theta-\cos3\pi/2n)\dots$

$$=2^{n-1}\begin{bmatrix}n\text{ odd}\\ \cos\theta\end{bmatrix}(\sin^2\pi/2n-\sin^2\theta)(\sin^2 3\pi/2n-\sin^2\theta)\dots$$

[stopping short of $\sin^2\pi/2$.

Proof.—$\cos n\theta$ is function of $\cos\theta$ which vanishes when $\cos\theta=\cos\pi/2n,\ \cos3\pi/2n,\dots$.

Then see *ALG.* § 3 (1) and to determine coefft. 2^{n-1} use § 24 *A.* (2). For second line couple first and last factors of first line, &c.

(2) $\sin n\theta$

$$=2^{n-1}\begin{bmatrix}n\text{ even}\\ \cos\theta\end{bmatrix}\sin\theta(\sin^2\pi/n-\sin^2\theta)(\sin^2 2\pi/n-\sin^2\theta)\dots$$

[stopping short of $\sin^2\pi/2$.

(3) $\cos n\theta-\cos n\phi=2^{n-1}\prod_{r=0}^{r=n-1}\{\cos\theta-\cos(\phi+2r\pi/n)\}$.

(4) $n=2^{n-1}\sin^2\pi/n\,.\,\sin^2 2\pi/n\dots,$

$1=2^{n-1}\sin^2\pi/2n\,.\,\sin^2 3\pi/2n\dots$

[stopping short of $\pi/2$.

(5) $\sin\theta=\theta\prod_{r=1}^{r=\infty}\{1-\theta^2/r^2\pi^2\}$,

$\cos\theta=\prod_{r=1}^{r=\infty}\{1-4\theta^2/(2r-1)^2\pi^2\}$.

Proofs.—(i) Prove first § 27 (2) or (1). Put $\theta=0$ and deduce (4). Divide (2) or (1) by corresponding results of (4). Put $n\theta=\alpha$ and $n=\infty$. This proof does not involve complex numbers. For more exact proof, *Hobson*, § 284.

(ii) $\sin\theta = 2\sin\frac{\theta}{2}\sin\frac{\theta+\pi}{2}$

$= 2^3 \sin\theta/4 \,.\, \sin(\theta+\pi)/4 \,.\, \sin(\theta+2\pi)/4 \,.\, \sin(\theta+3\pi)/4$

$= 2^{n-1}\sin\theta/n \,.\, \sin(\theta+\pi)/n \ldots \sin(\theta+\overline{n-1}\,\pi)/n$ [n a power of 2].

Couple second and last terms. Deduce § 27 (2), &c. This proof does not involve complex numbers. Remember uncoupled term, $\cos\theta/n$.

(iii) From § 27 (6). Write 2θ for θ. Put $x = 1$. Deduce $\pm\sin n\theta = 2^{n-1}\sin\theta\sin(\theta+\pi/n)\ldots$. Couple second and last terms. Deduce § 27 (2) with ambiguity. Put $\theta = 0$ and deduce § 27 (4), thus settle ambiguity. Rest as before.

(6) $x^{2n} - 2x^n\cos n\theta + 1$

$= \prod_{r=0}^{r=n-1}\{x^2 - 2x\cos(\theta + 2r\pi/n) + 1\}.$

Proofs.—(i) Solve for x^n and use De Moivre.

(ii) If u_n = L.H. prove $u_n - 2xu_{n-1}\cos\theta + x^2 u_{n-2}$ divisible by u_1. Hence by induction prove u_n divisible by u_1. Hence, &c.

(7) $x^n - 1 = (x^2-1)\prod_{r=1}^{r=\frac{1}{2}(n-2)}\{x^2 - 2x\cos 2r\pi/n + 1\}$ (n even)

$= (x-1)\prod_{r=1}^{r=\frac{1}{2}(n-1)}\{x^2 - 2x\cos 2r\pi/n + 1\}$ (n odd)

$x^n + 1 = \prod_{r=0}^{r=\frac{1}{2}n-1}\{x^2 - 2x\cos(2r+1)\pi/n + 1\}$ (n even)

$= (x+1)\prod_{r=0}^{r=\frac{1}{2}(n-3)}\{x^2 - 2x\cos(2r+1)\pi/n + 1\}$ (n odd)

(8) $\sin n\vartheta = 2^{n-1}\sin\theta\sin(\theta+\pi/n)\ldots,$ n factors.

$\cos n\theta = 2^{n-1}\sin(\theta+\pi/2n)\sin(\theta+3\pi/2n)\ldots.$ n factors.

Note.—If the first of these is thought of as $2\pi/2n, 4\pi/2n, \ldots$ the second is in a sense complementary to it, completing the series $\pi/2n, 2\pi/2n, 3\pi/2n, \ldots$.

28. Deductions from Factor Results.

(1) $\frac{1}{1^2} + \frac{1}{2^2} + \ldots \text{ to } \infty = \frac{1}{6}\pi^2.$

Similarly

$\sum_1^\infty 1/r^4 = \frac{1}{90}\pi^4, \sum_1^\infty 1/(2r-1)^2 = \frac{1}{8}\pi^2, \sum_1^\infty 1/(2r-1)^4 = \frac{1}{96}\pi^4.$

Proof.—Use § 27 (5) and series for $\sin\theta$. Take logs and equate coefficients.

(2) Series for $\log(\sin\theta/\theta)$, $\log\cos\theta$, $\cot\theta$, $\operatorname{cosec}\theta$, $\tan\theta$, $\operatorname{cosec}^2\theta$, $\sec^2\theta$, &c. See *DIFF. CALC.* § 7 (6).

Proof.—Use § 27 (5). Take logs and differentiate (or $\theta+h$ for θ and equate coefficients of h). The result for $\operatorname{cosec}^2\theta = \sum_{r=-\infty}^{r=\infty} 1/(\theta+r\pi)^2$ gives interesting series for special values of θ.

(3) Summation of $\sum_{r=1}^{r=\infty} \frac{1}{r^2+x^2}$, ... $\sum_1^\infty \left(\frac{1}{r^2+x^2}\right)^n$.

Use § 27 (5). Put $\theta = \pi x i$. Take logs. Differentiate. To proceed from $\Sigma(r^2+x^2)^{-1}$ to $\Sigma(r^2+x^2)^{-2}$, differentiate result of $\Sigma(r^2+x^2)^{-1}$ and divide by x.

(4) Factorization of $\cosh n\phi - \cos n\theta$.

Use § 27 (6) and put $x = e^{\phi}$.

(5) Partial Fraction results are obtained from, e.g., § 27 (3) by taking logs and differentiating, or by the ordinary methods of P.F.

Result: $$\frac{n\sin n\theta}{\cos n\theta - \cos n\phi} = \sum_0^{n-1} \frac{\sin\theta}{\cos\theta - \cos(\phi + 2r\pi/n)}.$$

(6) *Properties of Circle.*

De Moivre: If $ABCD\ldots$ be a regular n-gon in circle, centre O, radius a, and P be a point distant x from centre, and $AOP = \theta$,

$$PA^2 \,.\, PB^2 \,.\, PC^2 \ldots = x^{2n} - 2a^n x^n \cos n\theta + a^{2n}.$$

Cotes: If in De Moivre $\angle AOP = 0$ and $\alpha, \beta, \gamma\ldots$ be midpoints of arcs AB, BC,...

$$PA \,.\, PB \,.\, PC \ldots = x^n - a^n$$
$$P\alpha \,.\, P\beta \,.\, P\gamma \ldots = x^n + a^n.$$

(7) *Wallis's Formula* $(2\,.\,4\,.\,6\ldots 2n)/[1\,.\,3\,.\,5\ldots(2n-1)]$ $= \sqrt{n\pi}$ when n is a large integer.

Note.—By the *Calculus of Finite Differences* it may be proved that

$$n! = \sqrt{(2\pi n)} \,.\, n^n \,.\, e^{-n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \ldots\right),$$

an easy method of finding an approximation to $n!$.

SPHERICAL TRIGONOMETRY

1. Spherical Geometry.

Triangles. (1) The Congruence Theorems with regard to equal triangles all hold, i.e. two triangles are equal in every respect if (α) A, b, $c = A'$, b', c', (β) a, b, $c = a'$, b', c', (γ) A, B, $c = A'$, B', c' or A, C, $c = A'$, C', c', (δ) a, b, $A = a'$, b', A' and B, B' either both less or both not less than a right $\angle$, also in certain cases which have no plane analogies, e. g. if A, B, $C = A'$, B', C', also if A, B, and area $= A'$, B', and area.

Proof.—The superposition proofs will generally apply (α to δ), but it may be necessary to replace a Δ by the equal (but not superposable) antipodal Δ.

(2) The angles at the base of an isosceles Δ are equal and conversely.

Note.—But while the angles of an equilateral Δ are equal they may have a value $> 60°$, e. g. all 90°, 'Quadrantal Triangle.'

(3) The greater of two angles of a Δ is subtended by the greater side and conversely.

(4) It is *not* true that the exterior angle of a Δ is greater than the interior and opposite angles, or that the sum of the three angles is $2R$.

(5) *Lexell's Theorem.* If the base BC and the area of a Δ ABC be given, the locus of A is a small circle through the antipodes of B and C.*

(6) Speaking generally, no theorems of Plane Geometry will hold in Spherical Geometry which depend for their proof on the properties of parallels, on the sum of the angles of a Δ, or on the ratios of lines.

* Cf. Euc. i. 37.

Small Circles.

(7) Given the base BC of a Δ and $E-A$ (where E is the excess), the locus of A is a circle through B and C.*

(8) If $\angle BAC$ be the $\angle$ in a semicircle, $\angle B+\angle C=\angle A$.†

(9) If a chord (great circle) through a point P meet a circle in A and B, then $\tan\frac{1}{2}PA\tan\frac{1}{2}PB$ is constant for all positions of the chord. ‡

Note.—Two tangents to small circle from external pt. are equal.

(10) A great circle *Radical Axis* (from any point on which tangents to two small circles are equal) exists, and a *Radical Centre* for three circles. All circles orthogonal to a *Coaxal System* pass through two *Limiting Points* on the Line of Centres, equidistant from the R.A.

(11) The relations of *Poles* and *Polars* can also be developed, and a theory of *Polar Reciprocation* exists.

Notes.—(1) Be careful not to confuse this sense of the word *pole* with the one more usual in Spherical Trigonometry.

(2) The following are examples of 'reciprocal theorems':

If the vertical angle and the perimeter of a Δ are given, the envelope of the base is a circle.

If the base and area of a Δ are given, the locus of the vertex is a circle.

Triangles.

(12) A self-polar circle (see § 1 (11) and *Note*) exists for every triangle. Its radius may be imaginary.

(13) A circle (*Hart's Circle*) analogous to the N.P.C. exists for every triangle. Its centre lies on the join of the orthocentre and the centroid (i. e. the intersection of the medians). It touches the in- and ex-circles and has a radius $\tan^{-1}(\frac{1}{2}\tan R)$.

Quadrilaterals.

(14) For any cyclic quadrilateral $A+C=B+D$, and for any circumcyclic quadrilateral $a+c=b+d$.

(15) [Analogous to Ptolemy's Theorem.] If x, y be the diagonals of a cyclic quadrilateral

$$\sin\frac{a}{2}\sin\frac{c}{2}+\sin\frac{b}{2}\sin\frac{d}{2}=\sin\frac{x}{2}\sin\frac{y}{2}.$$

* Cf. Euc. iii. 21. This is really a part of Lexell's Theorem.

† Cf. Euc. iii. 31.

‡ Cf. Euc. iii. 35, 36, 37.

(16) *Remy's Theorem.* If z be distance between midpoints of diagonals,

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \frac{x}{2} \cos \frac{y}{2} \cos \frac{z}{2}.$$

Transversals, &c.

(17) If in a triangle ABC a great circle DEF cuts the sides in D, E, F, then

$$\frac{\sin BD}{\sin DC} \cdot \frac{\sin CE}{\sin EA} \cdot \frac{\sin AF}{\sin FB} = 1.$$

[Cf. *GEOM.* § 4.]

(18) If AD, BE, CF be concurrent, the same relation holds (paying no respect to sign).

(19) *Conversely,* if this relation holds,

AD, BE, CF are concurrent if 1 or 3 of the points D, E, F lie *within* the sides;

D, E, F are collinear if 0 or 2 points do so.

Anharmonics.

(20) The A.R. of 4 points A, B, C, D is

$$\frac{\sin AB}{\sin BC} \cdot \frac{\sin CD}{\sin DA} = \{ABCD\}.$$

Note.—Strict attention must be paid to sign here, $AD = -DA$.

(21) The row is harmonic when this A.R. $= -1$.

(22) In this case $\tan AB$, $\tan AC$, $\tan AD$ are in H.P. So also $\tan DA$, $\tan DB$, $\tan DC$.

(23) The A.R. of the pencil formed by joining O to A, B, C, D is

$$\frac{\sin AOB}{\sin BOC} \cdot \frac{\sin COD}{\sin DOA} = \{O.ABCD\} = \{ABCD\}.$$

2. Spherical Mensuration.

(1) Arc of small circle = corresponding arc of equator × cosine of latitude.

(2) Area of lune $(\angle A) = 2Ar^2$.

(3) Area of triangle $= (A + B + C - \pi) r^2$,

$= Er^2$. [See § 8.]

Note.—The angles must be measured in radians.

(4) Area of small circle, radius $\rho = 2\pi r^2 (1 - \cos \rho)$.

3. The Subsidiary Triangles.

(1) *The Supplemental Triangle.* Any theorem in Sph. Trig. remains true if we keep one side and the opposite angle unchanged and write for everything else its supplement.

Note.—Produce AB, AC to meet in A'. $A'BC$ is the supplemental triangle. There are three such.

(2) *The Polar Triangle.* Any theorem in Sph. Trig. remains true if we write for each angle the supplement of the opposite side, and for each side the supplement of the opposite angle.

Note.—If A_1 be a pole of BC, &c., $A_1B_1C_1$ is a polar triangle. There are eight such triangles. For one of them the sides are $\pi-A$, $\pi-B$, $\pi-C$, and the angles are $\pi-a$, $\pi-b$, $\pi-c$. Three others are supplemental to this one. The other four are antipodal to these four.

4. Relations between Sides and Angles.

[*Note.*—These are most easily learnt in immediate connexion with the corresponding plane formulae, which are accordingly printed in parallel columns.]

Plane.	*Spherical.*
(1) $a^2 = b^2 + c^2 - 2bc\cos A$, &c.	$\cos a = \cos b\cos c + \sin b\sin c\cos A$, &c
(2) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$	$\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$
$= \dfrac{2\Delta}{abc}$.	$= \dfrac{2\delta}{\sin a\sin b\sin c}$,
where Δ	where δ
$= \sqrt{s(s-a)(s-b)(s-c)}$.	$= \sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}$
(3) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$,	$\sin\dfrac{A}{2} = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$,
$\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$,	$\cos\dfrac{A}{2} = \sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}$,
$\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.	$\tan\dfrac{A}{2} = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$, &c

Plane.

(4) *Note.*—The spherical formulae giving the sides in terms of the angles, which have no useful analogues *in plano*, must be found from (3) when required and *not* remembered. They are added for the sake of reference only.

(5) *Napier's Analogies.*

$$\tan\frac{B-C}{2} = \frac{\frac{(b-c)}{2}}{\frac{(b+c)}{2}}\cot\frac{A}{2},$$

$$\tan\frac{B+C}{2} = \cot\frac{A}{2}.$$

(6) *Gauss's Analogies.*

$$\frac{a}{2}.\sin\frac{B-C}{2} = \frac{b-c}{2}.\cos\frac{A}{2},$$

$$\frac{a}{2}.\cos\frac{B-C}{2} = \frac{b+c}{2}\sin\frac{A}{2}, \text{ \&c.}$$

Spherical.

$$\sin\frac{a}{2} = \sqrt{-\frac{\cos S\cos(S-A)}{\sin B\sin C}},$$

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},$$

$$\tan\frac{a}{2} = \sqrt{-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}, \text{ \&c.}$$

$$\tan\frac{B-C}{2} = \frac{\sin\frac{(b-c)}{2}}{\sin\frac{(b+c)}{2}}\cot\frac{A}{2},$$

$$\tan\frac{B+C}{2} = \frac{\cos\frac{(b-c)}{2}}{\cos\frac{(b+c)}{2}}\cot\frac{A}{2}, \text{ \&c.}$$

The Polar Transformation (§ 3 (2)) of these gives two other formulae *not* to be remembered:

$$\tan\frac{b-c}{2} = \frac{\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}}\tan\frac{a}{2},$$

$$\tan\frac{b+c}{2} = \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}\tan\frac{a}{2}.$$

$$\sin\frac{a}{2}\sin\frac{B-C}{2} = \sin\frac{b-c}{2}\cos\frac{A}{2},$$

$$\sin\frac{a}{2}\cos\frac{B-C}{2} = \sin\frac{b+c}{2}\sin\frac{A}{2},$$

$$\cos\frac{a}{2}\sin\frac{B+C}{2} = \cos\frac{b-c}{2}\cos\frac{A}{2},$$

$$\cos\frac{a}{2}\cos\frac{B+C}{2} = \cos\frac{b+c}{2}\sin\frac{A}{2}, \text{ \&c.}$$

Notes.—(1) As the plane formulae here given are not usually learnt in Plane Trig. it should be noted that the Gauss Formulae may be easily derived from the Napier Formulae by equating the 'numerators'

on both sides, thus $\sin\frac{B-C}{2}\,.\,[\ ] = \sin\frac{1}{2}(b-c)\cos\frac{A}{2}$, and filling up the [ ] by analogy, arguing, 'To B on L.H. corresponds b on R.H., $\therefore$ to $\frac{A}{2}$ on R.H. corresponds $\frac{a}{2}$ on L.H.,' using *sine* in this term in the Napier Sine Formula, *cosine* in the Cosine Formula. This mnemonic does *not* apply to the Polar Napiers which give no fresh results.

(2) The Polar Napiers which are *not* memorized may be easily written down from the above Gauss results. Dividing the first by the third we get the first Polar Napier.

(3) The Plane analogues of any Spherical formula may be deduced by putting $a = \alpha/r$, $b = \beta/r$, $c = \gamma/r$, expanding sines, &c., in series and making $r = \infty$.

(4) In Solution of Triangles it must be remembered that the case ABa is ambiguous, as well as the case a, b, A.

5. Relation connecting Four Consecutive Parts.

Any four parts which are consecutive as we advance round the triangle (e.g. *AcBa*, *cAbC*, &c.) are connected by the following relation:

Product of cosines of mid. parts = cot ext side . sin mid side − cot ext angle . sin mid angle.

Note.—This important formula must be thoroughly practised. The printing indicates the mnemonic by which it is readily retained. As an example take *AcBa*. Here a is the 'extreme', 'exterior' side and c is the middle side, A is the 'ext.' angle and we have

$$\cos c\cos B = \cot a\sin c - \cot A\sin B.$$

6. Right-angled Triangles.

(1) *Napier's Rules.* A right-angled triangle contains 5 parts (for C is known). To write a formula connecting any three of these note that there is one of the three to which the other two are either 'adjacent' or 'opposite'. Further, *mentally* write *for every part its complement, except for those parts which run up to the right angle* (and may therefore by way of mnemonic be considered to have already enough to do with $\frac{1}{2}\pi$), and then apply the formula

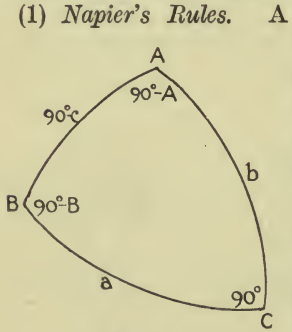


Fig. 4.

sin mid. = (*prod.*) cos *opp.* = (*prod.*) tan *adj.*

Notes.—(1) In full : sin middle part = product of cosines of opposite parts = product of tangents of adjacent parts.

(2) This is one of the most remarkable mnemonics ever invented, but in applying it it is essential that no additional figure should be used (as some of the text-books recommend). The triangle must be made to furnish its own figure mentally, and thorough practice with all the ten possible combinations of the sides is needed. Unless these precautions are adopted it is probably wisest to learn the formulae separately, and the student will then find that in the formula he wants he has forgotten which angle is taken as the right angle.

(2) In solving right-angled triangles we have an 'Ambiguous Case' when a and A are given (or b, B), namely one of the supplemental triangles.

7. Related Circles.

Plane.	*Spherical.*
(1) *Circumcircle.*	
$R = \dfrac{\frac{a}{2}}{\cos\left(\frac{\pi}{2} - A\right)}$	$\tan R = \dfrac{\tan\frac{a}{2}}{\cos(S-A)}$
$= \dfrac{2 \cdot \frac{a}{2} \cdot \frac{b}{2} \cdot \frac{c}{2}}{\Delta}$.	$= \dfrac{2 \sin\frac{a}{2} \sin\frac{b}{2} \sin\frac{c}{2}}{\delta}$.
(2) *Incircle.*	
$r = \dfrac{\Delta}{s}$	$\tan r = \dfrac{\delta}{\sin S}$.
$= a \cdot \dfrac{\sin\frac{B}{2} \sin\frac{C}{2}}{\cos\frac{A}{2}}$.	$= \sin a \dfrac{\sin\frac{B}{2} \sin\frac{C}{2}}{\cos\frac{A}{2}}$.
(3) *Excircle.*	
$r_1 = \dfrac{\Delta}{s-a}$	$\tan r_1 = \dfrac{\delta}{\sin(s-a)}$
$= a \dfrac{\cos\frac{B}{2} \cos\frac{C}{2}}{\cos\frac{A}{2}}$.	$= \sin a \cdot \dfrac{\cos\frac{B}{2} \cos\frac{C}{2}}{\cos\frac{A}{2}}$.

Notes.—(1) These formulae are all so easy to find that it is a question whether to learn them is worth the risk of error.

(2) In Spherical Geometry there are excircumcircles, the circumcircles of the supplemental triangles (R_1, R_2, R_3).

(4) Just as in Plane Trigonometry [*TRIG.* § 12 (5)] we have for the contacts of the incircle $\mathbf{B}D = s - \mathbf{b}$, $\mathbf{C}D = s - \mathbf{c}$, and that the contacts of the A excircle with AB, AC are distant $\mathbf{s}$ from A.

Note.—Hence $\tan r = \sin(s-a) \cdot \tan\frac{A}{2}$. [Cf. *TRIG.* § 12 (2).]

$$\tan r_1 = \sin s \cdot \tan\frac{A}{2}.$$ [Cf. *TRIG.* § 12 (3).]

(5) If O be the circumcentre $\angle OBC = S - A$.

(6) $2\tan R = \cot r_1 + \cot r_2 + \cot r_3 - \cot r$.

8. Spherical Excess.

(1) *Lexell's (Cagnoli's) Formula*:

$$\sin\tfrac{1}{2}E = \delta/\left(2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\right).$$

(2) *Lhuilier's Formula*:

$$\tan\tfrac{1}{4}E = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}.$$

Note.—These correspond to $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

The following may be added for reference:

(3) *Euler's Formula*: $\cos\frac{E}{2} = \dfrac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}$.

9. Tetrahedra.

(1) The geometry of a tetrahedron is often best dealt with by regarding it as inscribed in a parallelepiped as shown. The essential rule in drawing such a figure is that each edge of the tetrahedron must be a face-diagonal of the parallelepiped. We have the following species of tetrahedra:

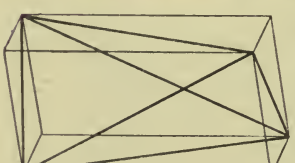


Fig. 5.

Tetrahedron.	*Parallelepiped.*
Isosceles (each edge = opposite edge).	Rectangular.
Orthogonal (each edge $\perp^r$ to opposite edge).	All edges equal, all faces rhombi.
Regular (all edges equal).	Cube.

(2) The lines joining middle points of opposite edges intersect in a point, bisect each other, and in isosceles tetrahedra are mutually perpendicular.

(3) Any plane through the centres of a pair of opposite edges divides the tetrahedron into two parts whose volumes are equal.

(4) The volume of the tetrahedron is one-third the volume of the generating parallelepiped.

(5) *Isosceles Tetrahedra.* The centroids of surface and volume coincide with one another and with the circumcentre, and lie on the joins of the centres of opposite edges.

(6) *Orthogonal Tetrahedra* are the most directly analogous to plane triangles. In them alone an orthocentre (H) exists ; the orthocentre, centroid, and circumcentre are collinear ; and there is a *Twelve Point Sphere* (analogous to the N. P. C.) which passes through the centroids of the faces, the orthocentres of the faces, and through four points of trisection of the joins of H to the four vertices. The radius of this sphere is $\frac{1}{3}$ radius of circumsphere, and the centroid and orthocentre are the centres of similitude of these spheres.

(7) *Regular Tetrahedra.* If a be an edge, the join of the centres of opposite edges is $\frac{1}{2}a\sqrt{2}$, the circumradius is $\frac{1}{4}a\sqrt{6}$, the inradius $\frac{1}{4}a\sqrt{2}$, the volume $\frac{1}{12}a^3\sqrt{2}$.

The faces intersect at $\angle \cos^{-1}\frac{1}{3}$.

The various centres (orthocentre, circumcentre, &c.) coincide at the centre of the generating cube.

10. Parallelepipeds. If a, b, c be edges intersecting at a point and $\alpha = \angle bc$, $\beta = \angle ca$, $\gamma = \angle ab$,

Diagonal through the point

$$= \sqrt{a^2 + b^2 + c^2 + 2bc\cos\alpha + 2ca\cos\beta + 2ab\cos\gamma}.$$

Volume

$$= abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}.$$

Proof.—Let diagonal make angles λ, μ, ν with edges. Project the diagonal and the broken line reaching from end to end of diagonal via the edges—

(1) on the diagonal, $D = a\cos\lambda + b\cos\mu + c\cos\nu$.

(2) on the edges in succession, $D\cos\lambda = a + b\cos\gamma + c\cos\beta$ and two similar equations.

Eliminate λ, μ, ν, &c.

11. POLYHEDRA.

(1) *Euler's Theorem.* If S be the number of solid angles in any polyhedron, F the number of its faces, E of its edges,

$$S + F = E + 2.$$

(2) *Regular or Platonic Solids.* There are only 5 :

Tetrahedron	made by	4	equilateral triangles.
Cube	,,	,,	6 squares.
Octahedron	,,	,,	8 equilateral triangles.
Dodecahedron	,,	,,	12 regular pentagons.
Icosahedron	,,	,,	20 equilateral triangles.

[See *MENS.* § 21.]

(3) *Archimedean Solids.* These have all their angles equal, and their faces regular polygons, which are not of the same species. They are 13 in number.

Truncated Tetrahedron :	4 triangles, 4 hexagons ; 2 hexagons and 1 triangle at each of 12 vertices.
Cuboctahedron :	8 triangles, 6 squares ; 2 triangles and 2 squares at each of 12 vertices.
Truncated Cube :	8 triangles, 6 octagons ; 2 octagons and 1 triangle at each of 24 vertices.
Truncated Octahedron :	6 squares, 8 hexagons ; 2 hexagons and 1 square at each of 24 vertices.
Small Rhombicuboctahedron :	8 triangles, 6 squares, 12 pentagons.
Great Rhombicuboctahedron :	8 triangles, 6 squares, 12 decagons.
Icosidodecahedron :	20 triangles, 12 pentagons.
Truncated Icosahedron :	20 hexagons, 12 pentagons.
Truncated Dodecahedron :	20 triangles, 12 decagons.
Snub Cube :	32 triangles, 6 squares.
Small Rhombicosidodecahedron :	20 triangles, 30 squares, 12 pentagons.
Great Rhombicosidodecahedron :	30 squares, 20 hexagons, 12 decagons.
Snub Dodecahedron :	80 triangles, 12 pentagons.

(4) *Kepler-Poinsot Solids.* These correspond to the 'star-polygons', of which the well-known 'pentagram' is an example. They are four in number, 3 dodecahedrons and an icosahedron.

GEOMETRY

Point-Systems.

1. *Euler's Theorem.* (A, B, C, D collinear.)

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

Note.—A particular case of Ptolemy's Theorem.

2. *Stewart's Theorem.* (A, B, C collinear, P any other point.)

$$AP^2 . BC + BP^2 . CA + CP^2 . AB = -BC \,.\, CA \,.\, AB.$$

3. *The Centroid Theorem* If G is the centroid of any masses m_1, m_2, ..., at points A_1, A_2, ..., and P is any point in their plane,

$$\Sigma[m_1 \,.\, A_1 P^2] = \Sigma[m_1 \,.\, A_1 G^2] + [\Sigma m_1] \,.\, PG^2.$$

Concurrence of Lines.

4. *Ceva's Theorem.* If D, E, F are points on the sides of a triangle, AD, BE, CF concur if

$$BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB,$$

and conversely.

Note.—An equivalent condition is $\sin BAD \,.\, \sin CBE \,.\, \sin ACF = \sin DAC \sin EBA \sin FCB$.

5. If D, E, F are points on the sides of a triangle the $\perp^{rs}$ at D, E, F concur if

$$BD^2 + CE^2 + AF^2 = DC^2 + EA^2 + FB^2,$$

and conversely.

6. If three lines through the vertices of a triangle concur so do also (*a*) their isogonal conjugates; (*b*) their isotomic conjugates.

7. Statical methods apply to many theorems involving middle points.

Ex. The joins of m. pts. of opp. sides of a 4[1] and the join of the m. pts. of the two diags. concur.

Collinearity of Points.

8. *Menelaus's Theorem.* If D, E, F are points on the sides of a triangle they are collinear if

$$BD \,.\, CE \,.\, AF = -DC \,.\, EA \,.\, FB,$$

and conversely.

Note.—The converse of this theorem (but not the direct theorem) is true for any n-gon, the sign being $(-)^n$.

9. *Pascal's Theorem* is sometimes of service [see *GEOM. CON.* § 33 (1)].

Ex. ABC is a $\triangle$ insed. in a circle; D and F are m. pts. of arcs BC, BA; P is *any* pt. on arc CA; PD, BC meet in L and PE, AB in N. Prove that LN passes through the incentre.

The Triangle.

[*Lettering of Triangle.* The following lettering is adopted for points connected with $\triangle ABC$:

D, E, F: feet of perpendiculars.

D', E', F': m. pts. of sides.

D_1, E_1, F_1: intersections with sides of bisectors of angles.

X, Y, Z: pts. of contact of inscribed circle.

O, I, H, G: circumcentre, incentre, orthocentre, centroid.

Ω, Ω'; K, N: Brocard points, symmedian point, nine-point centre.]

10. *General Properties.* If the base BC is divided in R so that $BR : CR = m : n$,

$$n\,AB^2 + m \,.\, AC^2 = n \,.\, BR^2 + m \,.\, CR^2 + (m+n)\,AR^2.$$

Notes.—(1) This is a particular case of § 3.

(2) If R be at m. pt. of BC, $AB^2 + AC^2 = 2AR^2 + 2BR^2$. [§ 12 (1).]

11. *Isogonals.* If X, Y be any pts. on lines isogonally conjugate:

(1) $XM \cdot YN = XP \cdot YQ$, and conversely.

(2) $MNPQ$ is cyclic.

(3) $MP \perp^r AY$.

(4) If three lines AX, BX, CX meet in a pt. their isogonals AY, BY, CY meet in the 'isogonal conj. point'.

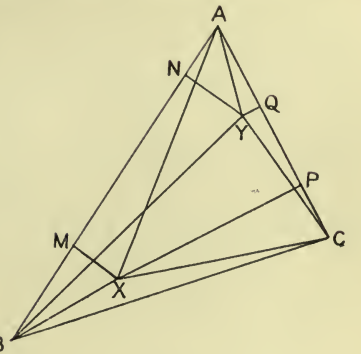


Fig. 6.

Note.—The following are Isogonal Conjugate Point-Pairs: Orthocentre, Circumcentre; Centroid, Symmedian Pt.; Incentre, Incentre.

12. *The Medians and Centroid* (G).

(1) $AB^2 + AC^2 = 2AD'^2 + 2BD'^2$. Hence AD' in terms of a, b, c.

(2) $D'G = \frac{1}{3} D'A$, &c.

(3) $\triangle BGC = \triangle CGA = \triangle AGB$.

(4) $\triangle D'E'F' \parallel\!\parallel\!\parallel \triangle ABC$.

(5) G is the centroid of wts. 1, 1, 1 at A, B, C; also of $\triangle D'E'F'$.

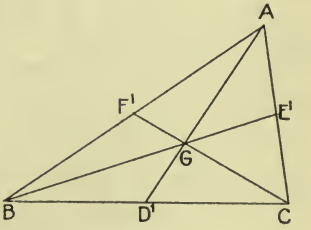


Fig. 7.

(6) The distances of G from the sides are as $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

(7) $x^2 + y^2 + z^2$ (areal co-ords. of P) is min. when P is at G.

13. *The Symmedians, and Symmedian (Lemoine) Point (K).*

(1) The A-symmedian bisects all antiparallels to BC.

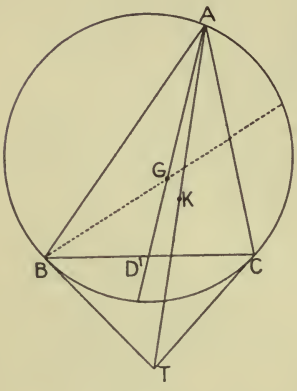


Fig. 8.

(2) If T is pole of BC w. r. $\odot ABC$, AT is the A-symmedian.

(3) The symmedian divides BC in ratio $c^2 : b^2$.

$$\Delta BKC : \Delta CKA : \Delta AKB = a^2 : b^2 : c^2.$$

Note.—Hence its length, by § 10.

(4) $D'K$ bisects $\perp^r AD$.

Note.—If $A = 90°$, K bisects $\perp^r AD$.

(5) K is centroid of wts. a^2, b^2, c^2 at A, B, C.

(6) Distances of K from the sides are in ratio $a : b : c$, and K is the centroid of the feet of these distances.

(7) $\alpha^2 + \beta^2 + \gamma^2$ (trilinear co-ords. of P) is min. when P is at K.

(8) If through K lines are drawn antiparallel to the sides of a triangle, the 6 points in which they meet the sides lie on a circle (**Cosine circle**) whose centre is K.

(9) If through K lines are drawn $\parallel$ to the sides of Δ, the 6 pts. in which they meet the sides lie on a $\odot$ (**Triplicate ratio circle**, *or* **Lemoine circle**), whose cent. is the m. pt. of OK. (O is the circumcentre.)

Note.—The names are due to the fact that these circles cut off from the sides segments whose lengths are:

Cosine Circle—$\cos A : \cos B : \cos C$.

Lemoine Circle—$a^3 : b^3 : c^3$.

14. *The Circumcircle.*

(1) $2R \cdot p_1 = bc$.

(2) If P is a pt. on the circumcircle and PL, PM, PN

$\perp^{rs}$ on the sides, LMN are collinear (**Pedal = Wallace = Simson Line**).

(3) If P and Q are on the circumcircle the $\angle$ between the pedal lines of P and $Q = \angle$ which PQ subtends on the circumcircle.

(4) If H is the orthocentre the pedal line of P bisects HP in a pt. lying on N. P. C.

(5) The pedal line of P is $\perp^{r}$ to the isogonal conjugate of the join of P to any angular pt. [§ 11 (3)].

15. *Incircle and Excircles.*

(1) $BD_1 : CD_1 = c : b$. Hence BD_1, CD_1.

$BD_1 . CD_1 + AD_1^2 = bc$. Hence AD_1. [Also by § 10.]

Similarly the external bisector AD_1'.

(2) L is circumcentre of $BICI_1$,

M is circumcentre of BI_2CI_3.

(3) $2D'M = r_2 + r_3$,

$2D'L = r_1 - r$,

$\therefore\ r_1 + r_2 + r_3 - r = 4R$.

(4) ΔAIZ

$\mid\mid\mid\ \Delta LBM$.

(5) $OI^2 = R^2 - 2Rr$,

$OI_1^2 = R^2 + 2Rr_1$.

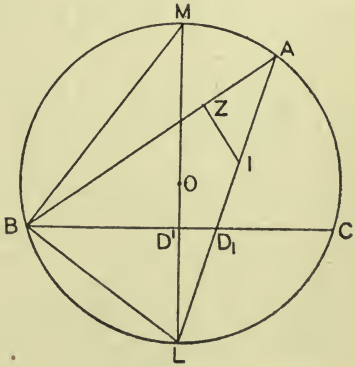


Fig. 9.

(6) If 2 circles with centres O, I and radii R, r are such that $R^2 - 2Rr = OI^2$, an infinite number of triangles can be inscribed in the first circle whose sides touch the second.

(7) $AZ = s - a$, $AZ' = s$.

16. *Orthocentre, Pedal Triangle, Polar Circle.*

(1) *BFEC, CDFA, AEDB* are cyclic.

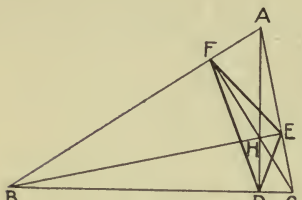


Fig. 10.

(2) *EF* is anti-∥ to *BC*.

(3) Δ's *AEF, BFD, CDE, ABC* are similar.

(4) The sides of the pedal Δ are $a\cos A$, $b\cos B$, $c\cos C$ and the angles $\pi - 2A$, &c.

(5) *H* is the incentre, *A*, *B*, *C* the excentres of *DEF*.

(6) $AH^2 = 4R^2 - a^2$.

(7) *Polar Circle.* *H* is the centre of the polar circle whose radius ρ is given by $\rho^2 = HA \cdot HD$. This circle (with regard to which Δ *ABC* is self-conjugate) is real only when one ∠ of the Δ is obtuse, e.g. in the figure the polar circle of *BHC* is real, its centre is *A*, its radius $\sqrt{AH \cdot AD}$.

(8) *Taylor Circle.* If *LL'* are the projections of *EF* on *BC*, *MM'* corresponding pts. on *CA*, *NN'* on *AB*, then *LL'MM'NN'* lie on the **Taylor Circle.**

Proof.—By Trig. prove $BL \cdot BL' = BN \cdot BN'$, ∴ *LL'NN'* cyclic, ∴ *LL'MM'NN'* cyclic (§ 25 (2) *Note*).

17. *Nine-Point Circle.*

(1) *O, G, H* are collinear.

(2) *N* bisects *OH*.

(3) A circle with centre *N* and rad. $\frac{1}{2}R$ passes through the 9 points *D, E, F*; *D', E', F'*; *A', B', C'* [the m. pts. of *HA, HB, HC*].

(4) *G* and *H* are the centres of similitude of this circle and the circumcircle.

(5) *Feuerbach's Theorem.* The N. P. C. touches the inscribed and the three escribed circles of a Δ.

Notes.—(1) It also touches the inscribed and three escribed circles of each of the Δ's *BHC*, *CHA*, *AHB*, 16 circles in all.

(2) To find W, the pt. of contact of N. P. C. and I. C. From D_1 draw D_1T tan. to I. C. Then $D'T$ meets I. C. again in W. [For lettering see above, § 10.]

(3) If L is m. pt. of arc which BC cuts off from N. P. C., XL passes through W. So also YM, ZN.

(4) W lies on the circumcircle of $D_1E_1F_1$.

(5) With a proper choice of signs $WD' \pm WE' \pm WF' = 0$, W being in this case the pt. of contact either with I. C. or with the excircles.

(6) For proof see § 38 (2). Also elegantly by Purser's Theorem [§ 26 (10)]. Apply to $\Delta\ D'E'F'$. $D'X = \frac{1}{2}(b \sim c)$, $E'F' = \frac{1}{2}a$, $\therefore\ \Sigma E'F' \cdot D'X = 0$, $\therefore\ \odot\ D'E'F'$ touches I. C. See also *ANAL. GEOM.* § 74, *Note*, and *TRIG.* § 13 (2).

(6) The circumcircle, N. P. C., and Polar Circle are coaxal.

18. *Brocard Points.*

(1) The marked angles are all equal [= the Brocard Angle ω].

(2) $\cot \omega = \cot A + \cot B + \cot C$.

(3) To construct Ω note that $\odot \Omega AC$ touches AB at A, $\odot\ \Omega BC$ touches AC at C.

(4) $A\Omega \cdot B\Omega \cdot C\Omega = A\Omega' \cdot B\Omega' \cdot C\Omega'$.

(5) $\begin{matrix}\Omega \\ \Omega'\end{matrix}$ are centres of similitude for ABC and an equiangular inscribed $\Delta\ \alpha\beta\gamma$, the angles equal to A, B, C being $\begin{matrix}\gamma, \alpha, \beta \\ \beta, \gamma, \alpha\end{matrix}$.

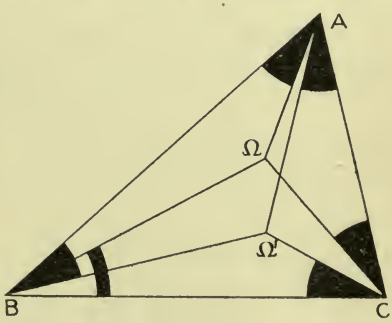


Fig. 11.

Note.—If the angles correspond in the order $\alpha\beta\gamma$ the circumcentre of ABC is the C. S. for any 2 pos^{ns} of $\alpha\beta\gamma$ and this circumcentre is the orthocentre of $\alpha\beta\gamma$. See § 19 (5).

(6) $\Omega\Omega' \perp^r OK$.

(7) Ω, Ω' lie on the circle on OK as diameter [**Brocard circle**], which also passes through the pts. $(A\Omega, B\Omega')$ $(B\Omega, C\Omega')$ $(C\Omega, A\Omega')$.

Two Triangles.

19. (1) *Desargues' Theorem.* When two triangles are in perspective (copolar, homologous) the corresponding sides intersect in three collinear points (axis of perspective, homology) and conversely.

(2) If ABC, $A'B'C'$ be two Δ's in perspective, and if $B'C'$ cuts AC, AB in the points X', X'' respectively, &c.,

$$\frac{AX'}{CX'} \cdot \frac{BY'}{AY'} \cdot \frac{CZ'}{BZ'} = \frac{AX''}{BX''} \cdot \frac{BY''}{CY''} \cdot \frac{CZ''}{AZ''},$$

and conversely.

Note.—To make the lettering clear note that

$$X' = (B'C', CA),\ Y' = (C'A', AB),\ Z' = (A'B', BC).$$
$$X'' = (B'C', AB),\ Y'' = (C'A', BC),\ Z'' = (A'B', CA).$$

(3) *Triangles directly similar.* If ABC, $A'B'C'$ be two directly similar triangles, the centre of similitude is the other intersection of the $\odot$'s BXB', CXC', where X is intersection of BC, $B'C'$.

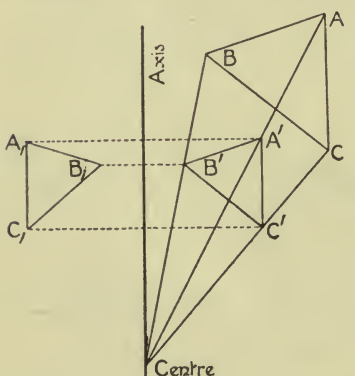


Fig. 12.

(4) *Triangles inversely similar.* If ABC, $A_1B_1C_1$ be two triangles inversely similar, the axis of similitude which bisects angles AOA_1, BOB_1, COC_1 is found by dividing AA_1, BB_1, CC_1 in the ratio of similitude, and O can be easily found by finding C'.

(5) *The 'Pivot Point'.* If a ΔPQR be inscribed in a ΔABC, the $\odot$'s AQR, BRP, CPQ pass through a common point O at which

$$BC \text{ subtends an } \angle (A+P), \text{ \&c.},$$
$$QR \quad ,, \quad ,, \quad \angle (\pi - A), \text{ \&c.},$$

so that if ABC, PQR be given in species, O is a fixed pt. w. r. either Δ, all positions of PQR are directly similar, and O is the C. S.

Note.—This very appropriate name, 'Pivot Point,' has been recently suggested by Mr. W. Gallatly for what other writers vaguely describe as the 'Point-O Theorem'. PQR ||| pedal triangle of O.

Harmonic Ranges and Pencils.

20. (1) *Def.* If C and D divide AB internally and externally in the same ratio, $\{ACBD\}$ is a *Harmonic Range.*

Notes.—(1) The ratio is sometimes given as (AB, CD) and sometimes, unfortunately, as $(ABCD)$.

(2) Note the particular cases (a) D at ∞, $AC = CB$, (b) A, B, C coincident; D anywhere.

(3) The symbol $\{ACBD\}$ is also used for the numerical value of the Anharmonic Ratio. See § 31 (1).

(2) If O be m. pt. of AB, $OC \cdot OD = OA^2$,
$$OC : OD = AC^2 : AD^2.$$

(3) If (AB, CD) is harmonic (CD, AB) is harmonic.

(4) $$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}; \quad \frac{2}{BA} = \frac{1}{BC} + \frac{1}{BD},$$
$$\frac{2}{CD} = \frac{1}{CA} + \frac{1}{CB}; \quad \frac{2}{DC} = \frac{1}{DA} + \frac{1}{DB}.$$

Note.—Sign must be carefully attended to.

(5) $\{ACBD\} = (AC/CB) \cdot (BD/DA) = -1.$

(6) For any Harmonic Pencil $\{O \cdot ACBD\}$ we have
$$\frac{\sin AOC}{\sin COB} \cdot \frac{\sin BOD}{\sin DOA} = -1.$$

(7) Any transversal cuts a H. P. in a H. R.

(8) The internal and external bisectors of an angle form a H. P. with the arms of the angle, and, conversely, if two conjugate rays of a H. P. be at rt. angles they bisect the angles between the other two.

(9) Given 3 rays of a H. P., to construct the fourth: through C on the middle of the 3 rays draw ACB bisected at C; a line through $O \parallel$ to this is the required fourth ray.

Note.—Another const. by § 23 (2).

21. *Involution.* **Def.** A series of pts. PP', QQ', ... on a line such that a point O exists for which

$$OP \cdot OP' = OQ \cdot OQ' = OR \cdot OR' = \ldots = k^2$$

constitutes an *involution* of which O is the *centre* and two points E and F such that $OE = OF = k$ the *foci* or *double pts.*

(1) The foci, EF, of an involution constitute a H. R. with every pair of points belonging to it, e.g. $\{PEP'F\} = -1$.

(2) Given 2 point-pairs of an involution, to determine foci:

Method i. Through PP' draw any $\odot$, through QQ' any $\odot$, let R. A. meet range in O; OF is the length of tan. from O to either circle.

Method ii. On PQ describe any $\triangle PQR$, on $P'Q'$ make $P'Q'R' \,|||\, PQR$. RR' meets range in O. OF is a mean proportional between OP and OP'.

(3) If $\{PP', QQ', RR'\}$ be in involution:

(*a*) $PQ' \cdot QR' \cdot RP' + P'Q \cdot Q'R \cdot R'P = 0$, and conversely.

(*b*) $\dfrac{PQ \cdot PQ'}{P'Q \cdot P'Q'} = \dfrac{PR \cdot PR'}{P'R \cdot P'R'}$, and conversely.

(4) If $O\{PP', QQ', RR'\}$ be a pencil in involution:

(*a*) $\sin POQ' \cdot \sin QOR' \cdot \sin ROP' + \sin P'OQ \cdot \sin Q'OR \cdot \sin R'OP = 0$, and conversely.

(*b*) $\dfrac{\sin POQ \cdot \sin POQ'}{\sin P'OQ \cdot \sin P'OQ'} = \dfrac{\sin POR \cdot \sin POR'}{\sin P'OR \cdot \sin P'OR'}$, and conversely.

HARMONIC PROPERTIES OF CIRCLE, POLE, AND POLAR.

22. (1) **Defs.** *Inverse Points.* P and P' on radius of circle such that $OP \cdot OP' = \text{rad.}^2$

Conjugate Points. Polar of each passes through the other.
Conjugate Lines. Pole of each lies on the other.

Notes.—(1) Inverse pts. are conjugate.
(2) The inv. of a pt. w. r. a st. line is its image in the line.

(2) If K and K' are inverse pts. and P be any pt. on the circle, $KP : K'P =$ const.

Conversely, if K and K' are fixed pts. and $KP : K'P$ = const., the locus of P is a circle (Circle of Apollonius) the extremities of whose diameter divide KK' in the given ratio.

(3) Any circle which passes through a pair of inverse pts. of another circle cuts that circle orthogonally.

(4) The line joining any pair of conjugate points is cut harmonically by the circle.

(5) If the polar of P passes through Q, the polar of Q passes through P.

If the pole of p lies on q, the pole of q lies on p.

(6) If the polars of P and Q intersect in R, the polar of R is PQ.

If the poles of p and q lie on r, the pole of r is pq.

Note.—Hence if P, Q be conj. pts. $\triangle PQR$ is self-conjugate and its orthocentre is at the centre of the circle.

(7) *Salmon's Theorem.* The distances of any two pts. P and Q from the centre of a circle are proportional to the distances of each from the polar of the other.

Quadrilaterals.

23. (1) *Def.* A complete *quadrilateral* (*tetragram*) / *quadrangle* (*tetrastigm*) is formed by 4 lines / points called *sides* / *vertices* which meet in / are joined by 6 points / lines forming 3 pairs of *opposite* *vertices* / *sides* (*connectors*), which pairs can be joined by / intersect in 3 other lines / points called *diagonals* (*harmonic diagonal pts.* (*har-*

lines) which form the sides / vertices of the *harmonic* (*diagonal*, *central*) *triangle* of the quadrilateral. / quadrangle. *harmonic* (*diagonal*, *monic pts.*)

(2) In a complete quadrilateral / quadrangle each pair of opposite vertices / sides forms a harmonic range / pencil with two of the angular points / sides of the diagonal triangle.

(3) The middle pts. of the 3 diagonals of a complete quadrilateral are collinear.

Note.—This line is sometimes called the '*diameter*' of the 4^1.

(4) The circumcircles of the 4 Δ's formed by any 4 lines pass through a common point.

(5) The orthocentres of the 4 Δ's formed by any 4 lines are collinear.

Note.—Prove by § 14 (4). This line is perpendicular to the 'diameter' of the 4^1.

(6) The circles described on the diagonals of a 4^1 as diameters are coaxal and their R. A. is the line given by (5).

Note.—The circle on the 3rd diagl. is the Circle of Similitude of those described on the other two.

(7) The three pairs of lines which join any point to the opposite vertices of a 4^1 are in involution.

(8) Any st. line is cut in involution by the three pairs of opposite connectors of any quadrangle.

(9) If A, B, C, D are any 4 pts. in a plane

$$AB \cdot CD + AD \cdot BC > AC \cdot BD$$

unless A, B, C, D lie in the order $ABCD$ on a circle or a st. line (in which case we have an equality).

Notes.—(1) Cf. § 1.

(2) A, B, C, D need not be in one plane.

Cyclic Quadrilaterals.

24. (1) The harmonic triangle of a quadrilateral circumscribed about quadrangle inscribed in a circle is self-conjugate (self-polar).

(2) The harmonic triangle of a quadrilateral circumscribed to a circle coincides with the harmonic triangle of the quadrangle formed by the pts. of contact.

This includes the following propositions :

(*a*) The internal diagls. of an escd. 4^1 and of the corresponding inscd. meet in a point.

(*b*) The third diagls. of an escd. 4^1 and of the corresponding inscd. are collinear.

(*c*) Any pair of opposite sides of an inscribed 4^1 intersect on a diagonal of the escd. 4^1.

(3) The 'diameter' [§ 23 (3)] of a circumscribed 4^1 passes through the centre of the circle.

(4) *Ptolemy's Theorem.*

$ac + bd = xy$. [See §§ 1 and 23 (9).]

(5) $$x^2 = \frac{(ac+bd)(bc+ad)}{ab+cd},$$

$$y^2 = \frac{(ac+bd)(ab+cd)}{bc+ad}.$$

(6) Inscribed Quadrilateral
$A + C = B + D.$
Escribed Quadrilateral
$a + c = b + d.$

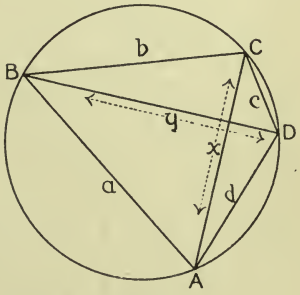


Fig. 13.

(7) For a cyclic quadrangle any straight line is cut in involution not merely by the three pairs of opposite connectors [§ 23 (8)], but also by the circle itself. **(Desargues' Theorem.)**

(8) For a circumcyclic quadrilateral the lines connecting any pt. with the vertices form a pencil in involution [§ 23

(7)] and the pair of tangents from the pt. also belong to the pencil.

(9) If a 4^1 can be inscribed in a ⊙ of radius R, and at the same time circumscribed to a ⊙ of radius r, and if d be the distance of their centres,

(*a*) $(R+d)^{-2}+(R-d)^{-2} = r^{-2}$ [and conversely].

(*b*) An infinite number of such 4^{1s} can be described.

Circles.

25. *Radical Axis.*

(1) If the R.A. of 2 ⊙'s, centres A, B, radii a, b, cut AB in O, $AO^2 - BO^2 = a^2 - b^2$.

(2) The R.A. of 3 ⊙'s, taken 2 and 2, intersect in a common point (**The Radical Centre**).

Note.—Hence if A_1A_2, B_1B_2, C_1C_2 be on the sides of a Δ and $B_1B_2C_1C_2$, $C_1C_2A_1A_2$, $A_1A_2B_1B_2$ be cyclic, then all 6 pts. lie on one and same circle.

Def.—The *Radical Circle* has centre at Rad. Centre and cuts all circles orthogonally.

(3) Difference of squares of tangents from any point P $= 2PN.AB$, where $PN = \perp^r$ from P on R.A.

Note.—The following are particular cases of R.A.

Concentric Circles.	R. A. the Line Infinity, L. P. the Circular Points.
Touching Circles.	R. A. tan. at pt. of cont. with which both L. P. coincide.
Line and Circle.	R. A. the line. To find L. P. use § 26 (1).
Point and Circle.	R. A. bisects $\perp^r$ dist. between pt. and inverse, which are the L. P.
Line and Point.	R. A. the line; L. P. the pt. and its image in the line.

26. *Coaxal Circles.*

(1) If ⊙'s, centre A, rad. a; B, b; C, c... are coaxal, then ABC... are collinear and

$$OA^2 - a^2 = OB^2 - b^2 = OC^2 - c^2 = \ldots = OL^2 = OL'^2,$$

where O is the pt. where the R.A. cuts the line of centres, and L, L' are 2 pts. (**Limiting Points**) lying on opposite sides of O on the line of centres.

Note.—The relation $BC \cdot a^2 + CA \cdot b^2 + AB \cdot c^2 = -BC \cdot CA \cdot AB$ is sometimes useful.

(2) If a line cut one circle of the system in PP', another in QQ', PQ, $P'Q'$ subtend at either L. P. angles which are equal or supplementary.

(3) A common tangent to 2 circles of the system subtends a rt. $\angle$ at L and L'.

(4) The L. P. are inverse w. r. each $\odot$ of the system.

(5) Any trv. cuts a coaxal system in involution.

Note.—If the trv. be line of centres, the foci of the involution are the L. P.

(6) The system of $\odot$'s through LL' themselves form a coaxal system (whose L. P. are imaginary if LL' are real) the members of which cut orthogonally every member of the given system, and have their centres on the R. A.

(7) If α, β, γ are 3 circles of a coaxal system, the tangents from any pt. of α to β and to γ are in a given ratio.

Conversely: If a point moves so that its tangents to 2 circles β and γ are in a given ratio, its locus is a circle α coaxal with β and γ.

(8) Note the particular case of (7) in which one of the circles becomes a L. P., e.g. in the figure PT/PA = const. as P moves on the outer circle.

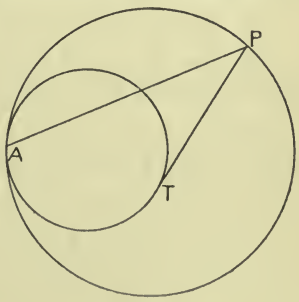


Fig. 14.

(9) The polars of a fixed pt. w. r. the $\odot$'s of a coaxal system are concurrent

(10) *Purser's Theorem.* From the vertices of a $\triangle ABC$ tans. AP, BQ, CR are drawn to a circle; if the sum of 2 of the rectangles $BC \cdot AP$, $CA \cdot BQ$, $AB \cdot CR$ is equal to the third, the given $\odot$ touches the $\odot ABC$.

Similitude.

27. (1) **Defs.** *Centres of Similitude* are 2 pts. S, S' dividing line of centres in ratio of radii.

Circle of Similitude is circle on SS' as diameter.

Circle of Antisimilitude, circle centre S (extl. C. S.), rad. $\sqrt{ST \cdot ST'}$. There is also an internal centre and circle of antisimilitude.

Notes.—(1) Lachlan reserves name C. S. for any pt. on circle of similitude and calls SS' **Homothetic Centres.**

(2) Particular Cases of C. S.:

Two Tangent Circles.	One C. S. at pt. of contact.
Circle and Line.	The two ends of diam. $\perp^r$ to line.
Circle and Point.	Coincide at the pt.
Line and Point.	Coincide at the pt.

(2) If in 2 circles $\parallel$ radii are drawn, the line joining their extremities passes through a C. S.

(3) If $ST'T$ be a com. tan., and $SQ'P'PQ$ a trv. cutting circles in Q', P', and P, Q, $SP \cdot SP' = SQ \cdot SQ' = ST \cdot ST'$.

(4) If $Sq'p'pq$ be a second trv., Pp, $P'p'$ intersect on the R. A.; so PT, $P'T'$ where $ST'T$ is com. tan.

(5) Every common tangent passes through a C. S.

(6) The 6 C. S. of 3 ⊙'s taken in pairs lie in threes on 4 lines called **Axes of Similitude.**

(7) If a circle touches 2 given circles, the line joining the pts. of contact passes through a C. S. of the two circles.

Note.—A particular case of (6).

(8) Two circles are similar w. r. any pt. on their circle of similitude, and conversely any pt. w. r. which 2 circles are similar lies on the circle of similitude.

(9) The tangents to 2 ⊙'s from any pt. on their circle of similitude are in the ratio of the radii. Hence:

(*a*) The two circles subtend the same angle at any pt. on the circle of similitude.

(*b*) The circle of similitude is coaxal with the given circles.

(10) The three circles of similitude of three given circles are coaxal.

(11) The circles of antisimilitude are also [cf. (9) (*b*)] coaxal with given circles.

(12) The tangents to 2 circles from any pt. on their circles of antisimilitude are in the ratio of sq. roots of radii.

Loci.

28. (1) A and B are *two* fixed pts., the locus of P, when

(*a*) $\angle APB$ = const., is 2 arcs of ⊙'s through A and B.

(*b*) $PA^2 + PB^2$ = const., is ⊙, cent. m. pt. AB.

(*c*) $PA^2 - PB^2$ = const., is st. line $\perp^r AB$.

(*d*) $PA : PB$ = const., is ⊙ dividing AB int. and ext. in ratio.

(*e*) $m \cdot PA^2 \pm n \cdot PB^2$ = const., is ⊙. [See § 3.]

(2) A, B, C, *three* fixed *collinear* pts., the locus of P, when

(*a*) $\angle APB = \angle BPC$, is circle through B dividing AC ext. in ratio AB/BC.

(*b*) $l \cdot PA^2 \pm m \cdot PB^2 \pm n \cdot PC^2$ = const., is ⊙. [See § 3.]

(3) A, B, C, D are *four* fixed *collinear* pts., the locus of P, when

(*a*) $\angle APB = \angle CPD$, is ⊙ on EF as diam., where E, F are foci of involution (AD, BC). [See § 21 (2), (1).]

(*b*) $l \cdot PA^2 \pm m \cdot PB^2 \pm n \cdot PC^2 \pm r \cdot PD^2$ = const., is ⊙.

(4) A, B, C, D are *four* fixed *coplanar* pts., the locus of P, when

(*a*) $\triangle PAB \pm \triangle PCD$ is const., is st. line.

(*b*) $m \cdot \triangle PAB \pm n \triangle PCD$ is const., is st. line.

(*c*) $\triangle PAB : \triangle PCD$ is const., is st. line through inters. of AB, CD.

(*d*) $l \cdot PA^2 \pm m \cdot PB^2 \pm n \cdot PC^2 \pm r \cdot PD^2$ is const., is circle.

(5) If PM, PN be drawn in fixed directions to meet two given lines, the locus of P, when

(a) $PM : PN$ = const., is st. line through the inters. of given lines.

(b) $m \,.\, PM + n \,.\, PN$ = const., is st. line.

(6) If $x, y, z, \ldots$ be $\perp^{rs}$ from P (or drawn from P in any given directions) to a series of st. lines, the locus of P is a st. line if $ax \pm by \pm cz \pm \ldots$ is const., where $a, b, c, \ldots$ are any constants.

(7) If the line joining a fixed pt. O to a variable pt. on a circle be divided in P in a fixed ratio, the locus of P is a $\odot$ and O is the C. S. of the two.

(8) Locus of P, when tans. to 2 given circles are in a fixed ratio, is $\odot$ coaxal with given circles. [§ 26 (7).]

(9) Locus of P, when tan. to given circle is in fixed ratio to distance from a given pt., is $\odot$ coaxal with $\odot$ and pt.

Note.—Hence, easily, if variable chord of $\odot$ subtend rt. $\angle$ at fixed pt., find locus of (a) its m. pt. *or* (b) foot of $\perp^r$ from fixed pt. upon it.

(10) Locus of P, when sq. of tan. to given circle $\propto \perp^r$ from P to given line, is $\odot$ coaxal with circle and line.

Envelopes.

29. (1) *Poncelet's Theorem*: If a polygon inscd. in a $\odot$ changes continuously so that each side but the last continues to touch a fixed circle of a system coaxal with the given $\odot$, then the last side also continues to touch a fixed $\odot$ of the system.

(2) *Bobillier's Theorem*: If the $\triangle ABC$ moves so that AB, AC touch fixed $\odot$'s, then BC also touches a fixed $\odot$.

Note.—Can be investigated by (4) (b), below.

(3) An envelope of a chord of a circle can sometimes be

determined by proving it to subtend a fixed angle at the circumference and thus to envelop a concentric circle.

(4) An envelope can often in one of two ways be reduced to a locus:

(*a*) *Infinitesimals.* Two consecutive positions of the moving line intersect in a point on the envelope. The locus of this point is the envelope.

(*b*) *Instantaneous Centres.* By this method a problem in envelopes can often be turned into one of loci.

Ex. AB, a line of fixed length, slides between two fixed trammels OA, OB. A is moving along OA, $\therefore$ turning instantaneously about some pt. I in $IA \perp^r OA$. So B is turning about I in $BI \perp^r OB$. Hence AB is turning about I as '*instantaneous centre*'. The envelope is the locus of N, the foot of $\perp^r$ from I on AB. This is not, however, one of the simple curves.

MAXIMA AND MINIMA. [See also § 39 (7), (8).]

30. (1) Given any 2 fixed pts. A, B, and any locus, to find the pt. C on the locus when

(*a*) Area ABC max. or min.	Tan. at $C \parallel AB$.
(*b*) $\angle ABC$,, ,,	Draw ⊙ thr. AB touching locus.
(*c*) $AC+BC$,, ,,	Tan. at C bisects $\angle ACB$ ext.
(*d*) $AC-BC$,, ,,	,, ,, ,, ,, int.
(*e*) AC^2+BC^2 ,, ,,	⊙ centre m. p. AB touches locus at C.

(2) *Fermat's Problem.* $PA+PB+PC$ is min. when P is the pt. at which all the sides subtend equal angles.

(3) For theorems involving $\Sigma(m_1 . A_1P^2)$ where m_1, m_2, m_3, ... are any constants, A_1, A_2, ... series of fixed pts., see § 3; e.g. $PA^2+PB^2+PC^2$ is min. when P is at centroid.

(4) Of all curves of given length the circle encloses greatest area.

(5) Of all curves of given area the circle has least perimeter.

(6) Given number of sides and area of a polygon, perimeter is min. if polygon is equilateral.

(7) Given number of sides and perimeter of a polygon, area is max. if polygon is equilateral.

(8) Given number of sides of a cyclic polygon, perimeter and area are max. if polygon is equilateral.

(9) Given number of sides of a circumcyclic polygon, perimeter and area are min. if polygon is equilateral.

(10) A convex polygon whose sides are given contains max. area when cyclic.

(11) If a st. line be divided into any number of parts, the continued product of all the parts is max. and the sum of their squares is min. when the parts are all equal.

Note.—The *Method of Infinitesimals* is by far the most powerful method available. Two examples are added.

Exx. (1) *Given any angle and any curve convex to this angle, then the tan. to the curve which forms with the sides of the angle a min.* $\triangle$ *is bisected at the point.*

(2) *Draw a line through P meeting any* 2 $\odot$*'s in A and B and such that* $AP \cdot PB$ *is min.* Take consecutive position, $\therefore AP \cdot PB = A'P \cdot PB'$, $\therefore AA'BB'$ cyclic, $\therefore$ we have to draw $\odot$ touching $\odot$'s in A and B and such that APB is st. line, $\therefore$ [§ 27 (7)] join C. S. of $\odot$'s to P meeting $\odot$'s in A and B.

Anharmonics. [See *A. GEOM.* § 66.]

31. (1) *Notation.*

A. R. of $A, B, C, D = \{ABCD\} = (AC, BD)$
$= (AB/BC) \cdot (CD/DA) = \lambda.$

A. R. of points arranged in other orders: $1/\lambda$, $1-\lambda$, $1/(1-\lambda)$, $(\lambda-1)/\lambda$, $\lambda/(\lambda-1)$.

For pencil A. R. $= \{O \cdot ABCD\}$
$= (\sin AOB/\sin BOC) \cdot (\sin COD/\sin DOA).$

(2) $\{ABCD\} = \{O \cdot ABCD\}$.

(3) To measure $\{O.ABCD\}$ numerically, draw any line $\parallel OD$ cutting OA, OB, OC in L, M, N, then

$$\{O.ABCD\} = -LM/MN.$$

(4) If $\{ABCD\} = \{A'B'C'D'\}$ and AA', BB', CC' pass through a pt.,

Then DD' also passes through it.

Cor. If $\{ABCD\} = \{AB'C'D'\}$ then BB', CC', DD', are concurrent.

(5) If $\{O.ABCD\} = \{O'.A'B'C'D'\}$ and $(OA, O'A')$ $(OB, O'B')$ $(OC, O'C')$ lie on a line,

Then $(OD, O'D')$ also lies on it.

Cor. If $\{O.ABCD\} = \{O'.A'B'C'D'\}$ and $OAA'O'$ is a st. line, then $(OB, O'B')$ $(OC, O'C')$ $(OD, O'D')$ lie on a st. line.

(6) If A, B, C, D are fixed pts. on a circle (conic) and P a variable pt. on it, then $\{P.ABCD\} = \text{const.}$

(7) If a, b, c, d are fixed tans. to a circle (conic) and p a variable tan. to it, then $\{(pa)(pb)(pc)(pd)\} = \text{const.}$

(8) *Pascal's Theorem*: If A, B, C, D, E, F are 6 pts. on a circle (conic), then the pts. (AB, DE) (BC, EF) (CD, FA) lie on a st. line.

Proof.—By (6) and (4) above.

(9) *Brianchon's Theorem*: If a, b, c, d, e, f are 6 tans. to a circle (conic), then the lines (ab, de) (bc, ef) (cd, fa) pass through a pt.

Proof.—By (7) and (5) above.

Notes.—(1) See also *G. CON.* § 33, and *A. GEOM.* § 65 (2).

(2) Note the simplest case of *Pascal's Theorem* when A, B, C lie on one line and D, E, F on another. This was known to Euclid.

32. *Homographic Ranges.* [See *G. CON.* § 32, *A. GEOM.* § 67.] (There is also a reciprocal theory of Homographic Pencils, &c., which can be readily constructed from the theory for ranges and is not given here.)

(1) *Definitions.* Any two ranges on the same or diff. lines are *homographic* when A. R. of any 4 points = A. R. of corresponding 4 points.

If $\{ABC\ldots\}$ (i) $= \{A'B'C'\ldots\}$, (ii) and O is pt. on (i) which corresponds to pt. at ∞ on (ii), O' pt. on (ii) corresponding to pt. at ∞ on (i), then $(AB, O\,\infty) = (A'B', \infty\, O')$; $\therefore\ AO \,.\, A'O' = BO \,.\, B'O' = PO \,.\, P'O'$ and O, O' are called **Centres** (Lachlan) or **Vanishing Points** (Russell) of the ranges.

If (i) and (ii) are on the same line, there will be 2 pts. of one range which coincide with corresponding pts. of the other. These are **Double Points** (Lachlan), **Common Points** (Russell), and are given by $OS \,.\, O'S = OA \,.\, O'A'$.

(2) *Construction of Homographic Range.* Given a range on one line, to construct a range homographic with it (a) on a second line, (b) on the same line.

(a) Let $\{ABC\ldots\}$ be the given range. Take any three pts. A', B', C' arbitrarily on the second line. The join of $(AB', A'B)$ $(AC', A'C)$ which also passes through $(BC', B'C)$ [see § 31 (8) note 2] is the *Homographic Axis* of the two ranges. Join $A'D$, meeting the Homographic Axis in δ, then $A\delta$ meets second range in required point D'.

To find the *Centres*, draw through $A' \parallel AB$, meeting Homographic Axis in I, then AI passes through O', &c.

The construction of (b) is also available.

(b) Join any pt. V outside the line to A, B, C.... Cut this pencil by any trv. whatever. Join any other pt. V' to the pts. thus obtained. The new pencil cuts the original line in a system homographic with $\{ABC\ldots\}$.

(3) *Construction of Common Points.*

Let $\{ABC\ldots\} = \{A'B'C'\ldots\}$, the ranges lying on the same st. line. To construct their common pts. take any $\odot$, let p be any pt. on it. Join pA, pB, pC, ... pA' ... meeting $\odot$ in a, b, c, ... a', ...

Construct Homographic Axis of $\{abc\ldots\}$ $\{a'b'c'\ldots\}$ [see (4) below]. Let this meet $\odot$ in H, H', then pH, pH' meet the line of the original ranges in their common points.

(4) If $\{ABC\ldots\} = \{A'B'C'\ldots\}$, the points lying on a circle (conic), then the pts. $(AB', A'B)$... all lie on a st. line, the **Homographic Axis (Axis of Homology)**.

(5) If through a pt. V outside a circle (conic) a pencil of rays be drawn cutting it in A, A'; B, B'; ..., then

$$\{ABC\ldots\} = \{A'B'C'\ldots\},$$

the Homographic Axis being the polar of V.

Proof.—§ 24 (1).

INVERSION. [See also *DIFF. CALC.* § 21 (2).]

33. *Inverse Terms* [O anywhere].

TERM.	INVERSE.
(1) St. line through pole O.	Itself.
(2) ,, not through O.	Circle through O. Such that orig. st. line is R.A. of the circle and the circle of inv.
(3) Two ∥ lines.	Circles touching at O.
(4) Curves inters^{g} in P, Q, R.	Curves inters^{g} in inv. points p, q, r.
(5) ,, touching at P.	Curves touching at inv. point p.
(6) ,, inters^{g} at $\angle\theta$.	Curves inters^{g} at $\angle -\theta$.
(7) Circle through O.	St. line $\perp^{r}$ diam. through O. Together with line at ∞.
(8) ,, not through O.	Circle. O a C. S. of the two; the inv. of O w. r. given ⊙ inverts w. r. O into cent. of inv. ⊙. If d and r be dist. of cent. from O and rad. of ⊙, then for inv. ⊙ $d' = R^2 d/(d^2 - r^2)$ $r' = R^2 r/(d^2 - r^2)$.
(9) R. A. of 2 ⊙'s.	Circle thr. O and int. of ⊙'s.
(10) C. S. of 2 ⊙'s.	Intersections of ⊙'s which touch the 2 inverses and pass through the pole.
(11) A circle and a pair of points (or figures) each inv. of other w. r. ⊙.	Inverse figures retain same relation.

34. *Inversion of Magnitude Relations*:

(1) $P'Q' = \dfrac{R^2}{OP \,.\, OQ} \,.\, PQ.$

(2) $\dfrac{\text{Square of common tan. of 2 circles}}{\text{Product of radii}}$ is unaltered.

(3) If t_1, t_2 be tans. from ext. pt. to ⊙'s radii r_1, r_2, then $(t_1^2/r_1) : (t_2^2/r_2)$ is unaltered.

(4) A. R. of 4 pts. is unaltered if O be on the line of the 4.

(5) If a range be inverted w. r. a pt. O not on it, it becomes a homographic circular range.

35. *Special Inverses.*

FIGURE.	INVERSE.	METHOD.
(1) Any circle.	(*a*) Itself.	R = tan. from O.
	(*b*) Circle with cent. at inv. of pt. A.	Take O at inv. of A w. r. ⊙. See § 33 (8). If $R = OA$ cent. inv. ⊙ is at A.
	(*c*) Any other circle.	O at C.S. of the two.
(2) Any 2 ⊙'s.	(*a*) Themselves.	O anywhere on R.A., R = tan. from O to ⊙'s.
	(*b*) St. lines.	O a pt. of intersection.
	(*c*) Circles of given sizes.	O anywhere on a particular ⊙ coaxal with given circles. See § 33 (8).
	(*d*) Equal circles.	Particular case of (*c*); O on either circle of antisimilitude.
(3) Any 2 ⊙'s which touch.	∥ st. lines.	O pt. of contact.
(4) Any 2 ⊙'s orthogonal.	⊙ and its diameter.	O on either circumf.
(5) Any 3 ⊙'s.	(*a*) Themselves.	O at rad. cent., R = tan. from O to any ⊙ of 3.
	(*b*) ⊙'s with centres collinear.	O on radical circle.
	(*c*) ⊙'s of given size.	O at intersections of ⊙'s given by 2 (*c*) (*d*) above.
	(*d*) Equal circles.	
(6) Coaxal circles.	(*a*) Themselves.	Cf. (2) (*a*) above.
	(*b*) Concurrent lines.	O at either inters. of ⊙'s.
	(*c*) Concentric circles.	O ,, ,, L. P. See § 26 (6).
	(*d*) Coaxal circles.	O anywhere; the L. P. inv. into the L. P.
(7) Circle and 2 pts. inverse w. r. it.	St. line, pt., image of pt.	O on the circle. See also § 33 (11).

36. *Inverses of Triangles and Quadrangles.*

FIGURE.	INVERSE.	METHOD.
(1) Vertices of any triangle, ABC.	(a) Vertices of equil. Δ. (b) ,, ,, Δ given species, $A'B'C'$.	Take O inside Δ ABC; $AA'B'B$ cyclic, &c. Deduce $BOC = A + A'$. Hence O. Cf. § 19 (5).
(2) Three sides of any triangle.	Three equal circles.	O at in- or ex-centres.
(3) Vertices of any quadrangle.	Vertices of $\parallel^m$.	Two positions for O; cf. solution to (1).
Vertices of cyclic quadrangle.	,, ,, rectangle.	

37. *Linkages.*

(1) *Peaucellier.*

$ABCD$ a rhombus,

$$OA = OC,$$

O fixed.

B and D describe inverse curves.

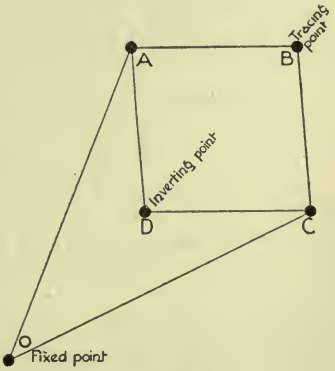


FIG. 15.

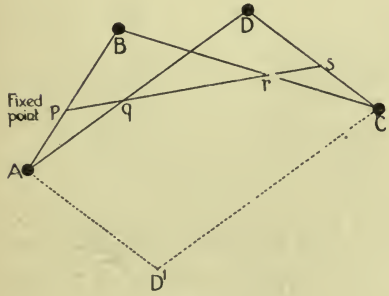


FIG. 16.

(2) *Hart's Contra-parallelogram.*

$ADCD'$ a $\parallel^m$, ABC image of $AD'C$ in AC, $ABCD$ is 'contra-parallelogram'.

$pqrs \parallel AC$.

When p is fixed, q and r describe inverse curves.

Note.—For a short account of Linkages see Carr's *Synopsis of Pure Mathematics* (5400–5449). Linkages may be constructed for trisecting an angle, for solving a cubic equation, for describing any integral algebraical curve, &c. Models, which are of great interest, can be easily constructed out of cardboard and drawing-pins.

38. *Theorems proved by Inversion.*

(1) *Casey's Theorem.* If 4 ⊙'s 1, 2, 3, 4 touch the same circle in the order 1234, and if $\overline{12}$ denotes a common tangent to 1 and 2 which is direct or transverse according as 1 and 2 are touched directly [i. e. 1 and 2 on same side] or transversely by the touching ⊙, then

$$\overline{12}\,.\,\overline{34} + \overline{14}\,.\,\overline{23} = \overline{13}\,.\,\overline{24}.$$

Notes.—Cf. Ptolemy's Theorem.

Proof.—Easily derived by inversion from Euler's Theorem (§ 1) but can be proved independently.

(2) *Feuerbach's Theorem* [§ 17 (5)].

(*a*) From Casey's Theorem for m. pts. of sides and incircle are 4 ⊙'s satisfying the conditions of Casey's Theorem.

(*b*) Take *O*, m. pt. of *BC*, which is on R. A. of I. C. and *A*-Ex- C., and rad. of inv. the tan. from *O* to I. C. *O* is on N. P. C. ; ∴ I. C. and Ex-C. inv. into themselves and N. P. C. into a line ; prove this line to be 4th common tan. of 2 ⊙'s ; (1) prove *D*, the ft. of ⊥^{r} from *A* to *BC*, to lie on this c.t. ; (2) prove it ⊥^{r} *ON*.

(3) *Hart's Theorem.* If the sides of a triangle be replaced by three circles, and four circles corresponding to the in- and ex-circles are drawn to touch them, the group of four is touched by a circle.

Problems on Triangles.

39. (1) *Construction of Triangles with given data.*

(*a*) a, A, $b : c$.	Locus of A, = const., intersects locus of $b : c$, = const., in A.
(*b*) a, B, $b : c$.	Cf. solution to (*a*).
(*c*) A, B, C and length of any line connected with △.	Proportional reduction of similar △ described about given line.
(*d*) p_1, p_2, p_3.	$ap_1 = bp_2 = cp_3$, ∴ △ is given in species.

(2) *To inscribe square (quadrilateral of given species) in given triangle.*

Describe $\triangle$ ||| given $\triangle$ about 4^1. Proportionate reduction.

(3) *To describe $\triangle ABC$ given in species, the point A being given and B and C lying* (a) *on given lines OB, OC,* (b) *on given circles.*

(*a*) Suppose done. Draw $AK \perp^r OB$. Make $\angle KAL = \angle BAC$. $CL \perp^r AL$. $\triangle AKB$ ||| ALC, $\therefore$ $AL:AK = AC:AB$, $\therefore$ AL known, $\therefore$ L known. From L reconstruct C, &c.

(*b*) Let K be cent. of B-circle. Join AK. Make $\triangle AKL$ ||| ABC. Thus L is known. With cent. L and rad. KB . (AC/AB) describe $\odot$. It cuts C-circle in C.

(4) *To describe $\triangle ABC$ given in species with vertices on three concentric circles.*

Let p, q, r be radii. Take any $\triangle ABC$ of species. Use § 28 (1) (*d*) to determine O so that $OA : OB : OC = p : q : r$. Proportionate reduction.

(5) *To describe $\triangle ABC$ given in species with vertices on three concurrent lines, and* (a) *one side of given length,* (b) *one side through given point.*

(*a*) Triangle is given in magnitude. Use § 28 (1) (*a*).

(*b*) Take any $\triangle$ of species. Use § 28 (1) (*a*) to draw three concurrent lines through vertices similar to given lines. Through given pt. draw $\parallel$ to one side of $\triangle$, &c.

(6) *To inscribe (or escribe) given $\triangle PQR$ to given $\triangle ABC$.*

To inscribe, determine 'pivot-point' [§ 19 (5)] for both triangles. With ABC-point-O as cent., rad. OP, describe $\odot$ cutting BC in P, &c.

(7) *To inscribe minimum $\triangle$ of given species in ABC.*

Determine 'pivot-point' [§ 19 (5)]. Draw OP, OQ, $OR \perp^r$ sides.

(8) *To circumscribe maximum $\triangle$ of given species about ABC.*

Determine 'pivot-point' [§ 19 (5)]. Draw $QR \perp^r OA$ thr. A, &c.

(9) *To inscribe in ABC a $\triangle PQR$ given in species and* (a) *one (all) side $\parallel$ given line (lines),* (b) *one side, QR, through given point X.*

(*a*) Draw any line $Q'R' \parallel$ given line. Make $P'Q'R'$ ||| ABC. Join AP' meeting BC in P, &c.

(*b*) Determine 'pivot-point' [§ 19 (5)]. ORX is known. Hence R [§ 28 (1) (*a*)].

(10) *To inscribe in ABC a Δ PQR whose sides pass through three given points α, β, γ.*

Through α draw any line αQR meeting AC, AB in Q, R. Join γQ, βR meeting BC in P_1', P_1. For a series of points so determined we have $\{P_1\} = \{R\} = \{Q\} = \{P_1'\}$. The common points [§ 32 (1)] of these homographic ranges determine P.

(11) *To circumscribe about ABC a Δ PQR whose vertices lie on any three given lines.*

This is the same problem as (10).

Note.—Exx. (10), (11) may be written : '*Given any 2 Δ's, to describe a Δ which is circumscribed to one and inscribed to the other.*'

(12) *Castillon's Problem. To inscribe in a given circle (conic) a Δ whose sides* (a) *pass through three given pts.*, (b) *pass two through given pts., the third $\parallel$ given line.*

(*a*) If the Δ in solution to (10) be replaced by a $\odot$, then using § 32 (5) the solution of (10) holds throughout.

(*b*) Is a particular case of (*a*) with third pt. at ∞.

Note.—Elementary solutions are possible (Smith and Bryant, *Euclid*, p. 408, &c.). The solution given above applies to an n-gon whose sides pass through n-points.

(13) *To circumscribe about a given circle a Δ whose sides lie on three given lines.*

Let p, q, r be the poles of the lines; inscribe Δ whose sides pass through p, q, r, &c.

(14) *To inscribe in ABC a Δ PQR whose sides subtend given angles α, β, γ at three given points L, M, N.*

Draw any line LQ_1 to AC. Make $\angle Q_1LR_1 = \alpha$, R_1 on AB; $R_1MP_1 = \beta$, P_1 on BC; and $Q_1NP_2 = \gamma$, P_2 on BC. Prove that, for various positions of Q_1, P_1 and P_2 describe homographic ranges, the common points of which determine P. Similar solution for polygons with n points and n angles.

(15) *To inscribe in a Δ ABC a rectangle of given area.*

A solution using a hyperbola is easy. Otherwise take any point P_1 in AC, draw $P_1N_1 \parallel BC$ to AB. Find $P_2M_2 \parallel AB$, P_2 on AC, M_2 on BC, and such that $P_1N_1 \cdot P_2M_2 =$ given area/sin B. P_1, P_2 describe homographic ranges whose common points give P, the vertex of the rectangle which lies on AC.

Examples (10)–(15) illustrate the **Method of False Positions** or **Method of Trial and Error**, perhaps the most powerful method for geometrical problems which exists.

Problems on Quadrilaterals.

40. (1) *To describe a square about a given quadrilateral ABCD.*

Draw AX perpendicular and equal to BD, CX is the direction of a side of the square. See also (2) below.

(2) *To describe a quadrilateral of given shape about a given quadrilateral ABCD.*

Let it be $KLMN$, the points being in order $KBLCMDNA$. Let $\odot KAB$ cut KM in R, $\angle ABR = \angle NKM$ and is known, so $\angle BAR$. Hence R is known. Similarly S by $\odot DCM$. From RS reconstruct figure.

(3) *To inscribe a quadrilateral of given shape in given quadrilateral ABCD.*

About a 4^1 of given shape describe a 4^1 similar to $ABCD$. Proportional reduction.

(4) *To construct a cyclic quadrilateral with given sides a, b, c, d.*

Suppose these sides represent AB, BC, CD, DA and that 4^1 is constructed. Make $\angle DCE = \angle ACB$, E on AD (prod.). Prove $DCE \parallel\!\!\parallel ABC$, $\therefore DE = ac/b$ and $AC : CE = b : c$. Hence reconstruct figure.

(5) *Castillon's Problem. To inscribe a quadrilateral in a given circle so that its sides should pass through four given points.*

See 39 (12), *Note.* A solution is also possible by **Inversion.**

(6) *To circumscribe a quadrilateral about a given circle so that its angles should lie on four given lines.*

Cf. 39 (13).

Problems on Circles.

[*Notation.* The following problem: *Describe a circle to pass through a given point* (**P**) *to touch a given circle* (**C**), *and to intersect a given line at angle* α ($\mathbf{L}_\alpha$) is given as $\mathbf{PCL}_\alpha$.]

41. *The Tangency Problems.*

We have *One Solution*: **PPP.**

Two Solutions: **PPL, PLL, PPC.**

Four Solutions: **LLL, PLC, PCC.**

Eight Solutions: **LLC, LCC, CCC.**

PPL. Join PP and produce to meet L, draw any tan. from this pt. to any ⊙ through PP, mark off on L line equal to this tan., &c.

PLL. *Sol.* i. Passes also through image of P in bisector of angle LL. *Sol.* ii. With centre any pt. on bisector of angle LL describe ⊙ touching LL. Join pt. (LL) to P meeting ⊙ in A and B. Join A (or B) to cent. of ⊙. Draw ∥ through P, &c.

PPC. Draw any ⊙ through PP meeting C in A and B. Let AB meet PP in O. Tan. from O to C, &c.

PLC. Draw diam. SA of $C \perp^r L$ and meeting L in B. Then [§ 27 (1) *Note* 2] S is C. S. of L and C. Join SP and take Q on it so that $SP \,.\, SQ = SA \,.\, SB$. Required circle passes through Q [§ 27 (7)]. Problem becomes *PPL*.

PCC. S a C. S. of CC, Q such that $SP \,.\, SQ$ = product of tans. from S to CC, then Q lies on required circle and problem becomes *PPC*.

LLC. Draw lines ∥ LL and distance from them = radius of C. Problem becomes *PLL*.

LCC. Cf. Sol. i to *CCC*. Reduces thus to *PLC*.

CCC. *Sol.* i. Draw ⊙'s concentric with C_2, C_3 and of same radii increased or diminished in each case by rad. of C_1. The problem becomes *PCC*.

Sol. ii. By Inversion from *CLL*.

Sol. iii (Casey). Let C_1, C_2, C_3 be the ⊙'s; S_1, S_2, S_3 their C.S.'s forming an Axis of Similitude; P_1 any pt. on C_1. Join P_1S_3 meeting C_2 in corresponding pt. Q_2, Q_2S_1 meeting C_3 in R_3, R_3S_2 meeting C_1 in p_1. Then P_1p_1 describe homographic ranges and either common point P of the ranges joined similarly to the C. S.'s determines Q and R the pts. of cont. of the tan. ⊙.

Sol. iv (Gergonne). Let O be the radical centre of the CCC, and LMN be poles of one of the 4 Axes of Similitude, then if OL meet its ⊙ in Pp, &c., the ⊙'s PQR, pqr touch CCC.

Notes.—(1) The problem is known as *Viète's Problem*, though it was proposed and perhaps solved by Apollonius.

(2) Sols. iii and iv provide alternative methods for many of the previous cases *LCC*, *LLC*, &c.

42. *Orthogonal Circles.* The problem of describing circles to cut two or more circles orthogonally depends on § 26 (6).

Diametral Circles. The problem of describing circles to cut two or more circles diametrally (i.e. to bisect the circumferences) depends on the fact that if A and B be the centres and a and b the radii, $AP^2 + a^2 = BP^2 + b^2$. Hence use § 28 (1) (*c*).

43. *Circles cutting others at given angles.*

$\mathbf{L_\alpha L_\beta}$. Let OA, OB be the lines. Take any pts. A, B. Make

$$\angle OAP = 90° - \alpha, \angle OBP = 90° - \beta.$$

Make PQ (in PB) $= PA$. Join AQ and produce to OB in S. Draw $SR \parallel BP$ to meet AP in R. OR is locus of centre. Hence easily $\mathbf{L_\alpha L_\beta P}$ (cf. *PLL* above), $\mathbf{L_\alpha L_\beta L}$, $\mathbf{L_\alpha L_\beta L_\gamma}$.

$\mathbf{L_\alpha L_\beta C_\gamma}$. *Sol.* i. By Inversion from $\mathbf{L_\alpha L_\beta L_\gamma}$.

Sol. ii. Let R on the $L_\alpha L_\beta$ locus be required centre and let required circle cut C in M and OA in A. Join RM, RA. Then $RMC = \gamma$. Make $ROQ = \gamma$, and cut off $OQ : OR = MC : MR = MC : AR$, $\therefore$ $OQ : MC = OR : AR$ = const. for species of OAR is known. Hence Q is a known point. But $QR : CR = QO : CM$ = const., $\therefore$ since Q and C are fixed, R lies on known circle.

$\mathbf{C_\alpha C_\beta C_\gamma}$ and all other cases may now be derived by Inversion, though other constructions are possible.

GEOMETRICAL CONICS

THE PARABOLA

1. Linear Properties (Tan. and Normal).

(1) $SL = SX = 2SA$.

(2) $SP = SG = ST = PM$.

(3) $NT = 2AN$.
$NG = 2AS$.

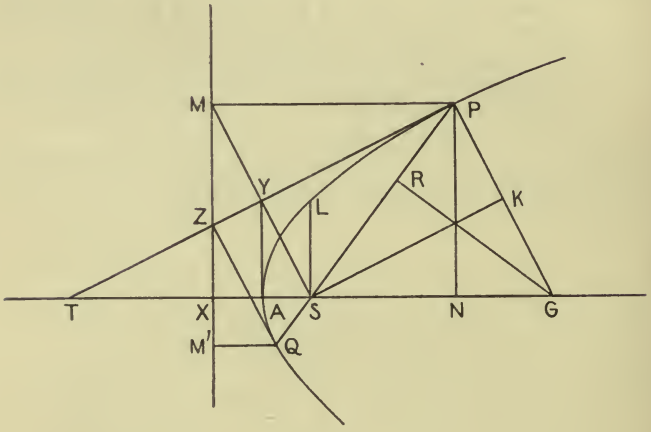


Fig. 17.

(4) $PN^2 = 4AS \cdot AN$.
$PG^2 = 4AS \cdot SP$.

(5) The complete normal at $P = 4ZQ$.

(6) $PR = SL$.

(7) SYM is a st. line; $TY = YP$, $SY = YM$; Y is on tan. at vertex; $SY^2 = SA \cdot SP$. $(p^2 = ar.)$

(8) The locus of K is a coaxal parabola, vertex S, latus rectum AS. The intersections of this parabola with a circle on SO as diameter easily give the points on the original parabola, normals at which pass through O, O being any point whatever.

(9) The algebraical sum of the ordinates of three conormal points is zero, and their circumcircle passes through A.

Proof.—Cf. *Note* to § 6 (4), and use § 1 (8).

2. Angular Properties (Tan. and Normal).

(1) The angles MPT, SPT, STP, SYA, NPG, PGR are all equal; MS bisects $\angle TSP$, $\therefore$ $\angle MSM'$ is rt. $\angle$.

(2) Δ's MYP, SYP, MYZ, SYZ, MPZ, SPZ, SYT, AYT, SAY, TPN, PNG, TPG, PRG are similar.

(3) (*a*) $\angle PSZ =$ rt. $\angle = \angle PZQ$.

(*b*) Tangents at end of a focal chord intersect at rt. angles on directrix.

(*c*) $SZ^2 = SP \cdot SQ$.

3. Chords.

(1) $AT^2 = AN \cdot AN'$.

Note.—True for any diameter $TP^2 = PV \cdot PV'$

[where chord $Q'Q$ meets diameter PT in T and QV, $Q'V'$ are ordinates. Cf. Fig. 19].

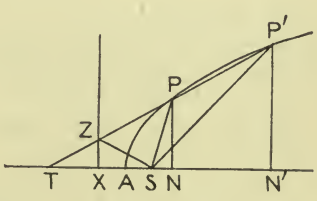


Fig. 18.

(2) SZ bisects exterior $\angle PSP'$.

(3) Semi-latus rectum is Harmonic Mean between the segments of any focal chord.

Note.—i.e. for a focal chord SP. $SQ = 4AS \cdot PQ$.

4. Diameters [marked angles are equal].

(1) Locus of centres of ∥ chords is st. line (*diameter*) ∥ to axis.

(2) Tangents at end of any one of these chords intersect on the diameter.

(3) Tangent at P is bisected at P (between TQ, TQ').

(4) Perpendicular from S on QQ' meets PV in directrix

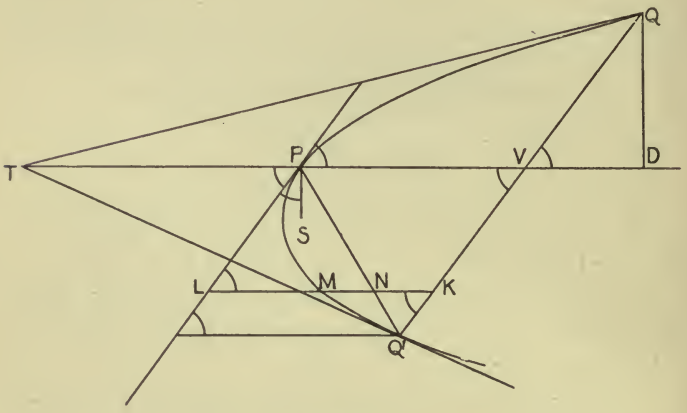


Fig. 19.

(5) Parameter [i.e. focal chord bisected by diameter] $= 4SP$.

(6) $TP = PV$.

(7) $QV^2 = 4SP \cdot PV$; $QD^2 = 4AS \cdot PV$.

(8) $LM : MN = PN : NQ'$.

(9) $QK \cdot KQ' = 4SP \cdot KM$.

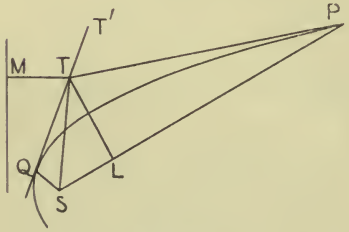


Fig. 20.

5. Two Tangents.

(1) (*Adams*) $SL = TM$ [$= SL'$].

(2) $\triangle STP \mid\mid\mid \triangle STQ$.

(3) $ST^2 = SP \cdot SQ$.

(4) $TP^2 : TQ^2 = SP : SQ$.

(5) $\angle T'TP = \frac{1}{2} \angle PSQ$.

6. Segments of Chords. [See also § 4 (8), (9).]

(1) If PP', QQ' are fixed in direction,

$(PO \,.\, OP')/(QO \,.\, OQ')$

is independent of O,

and $= Tp^2/Tq^2$,

$= Sp/Sq$,

= ratio of focal chords $\parallel PP'$, QQ'.

(2) If any Δ has its base $\parallel$ to the axis, the squares of its sides are in the ratio of the $\parallel$ focal chords.

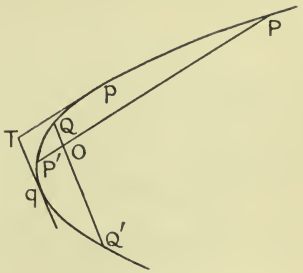


Fig. 21.

(3) $TE^2 = TA \,.\, TB$.

Note.—Hence, easily, § 3 (1).

(4) If a circle meets a parabola in four points A, B, C, D, the chords AB, CD are equally inclined to the axis.

Note.—Hence diameters of AB, CD are equidistant from axis, $\therefore$ algebraic sum of ordinates A, B, C, $D = 0$.

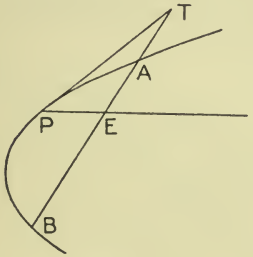


Fig. 22.

7. Three Tangents.

(1) Circle pqr passes through S.

(2) Orthocentre of pqr lies on the directrix.

(3) $Rp : pq = pQ : Qr$

$= qr : rP$.

(4) If a variable tangent cut the three fixed tangents in L, M, N, then

$LM : MN =$ const.

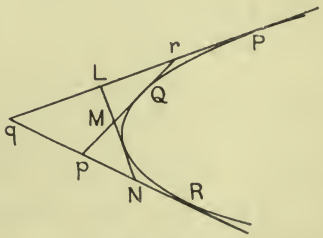


Fig. 23.

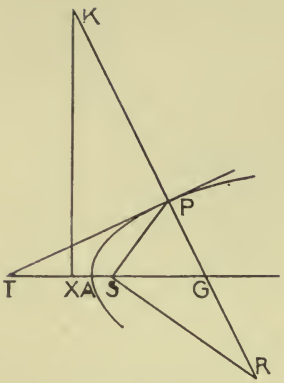


Fig. 24.

8. Curvature.

(1) Common chord of curvature at $P = 4\,PT$.

(2) Focal chord of curvature at $P = 4\,SP$.

(3) *Radius of Curvature* at P

$= PG^3/(2\,a^2) = 4\,SP^2/SY$
$= 2\,PK$.

Notes.—(1) Easily recalled by $\rho = rdr/dp$ from $p^2 = ar$.

(2) To find c. of curvature at P: join SP, draw $SR \perp^r SP$ meeting PG in R, then R is m. pt. of radius of curvature, *or* use radius of curvature $= 2\,PK$.

9. Areas.

(1) $\triangle\, Tqq' = \frac{1}{2} \triangle\, PQQ'$.

(2) Area of parabola $QPQ' = \frac{2}{3} \triangle\, TQQ'$.

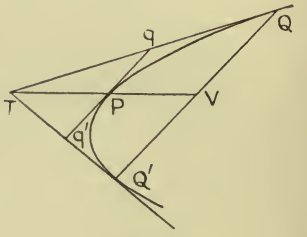


Fig. 25.

10. Drawing of Tangents. To draw the tangent at T:

i. *T on curve.* Use $SP = ST$.

ii. *T off curve.*

(1) Use Adams [§ 5 (1)].

[S cent., rad. $SL = TM$, describe ⊙. From T draw tans. to this.]

(2) T cent., TS rad., ⊙ cuts directrix in MM'. Diams. through M, M' meet curve in pts. of contact.

(3) Or, in (2), lines bisecting SM, SM' at rt. angles meet curve in pts. of contact.

(4) Circle on ST as diam. meets tan. at vertex in Y, Y', &c.

11. Construction of Parabola under given conditions.

[N.B.—Axis = 2 pts., direction of axis = 1 pt., focus = 2 pts., directrix = 2 pts.]

Data.	*Construction.*
(1) P, Q, S.	Draw circles, centres P, Q, radii PS, QS. Directrix is com. tan. (Two solutions.)
(2) 2 tans., S.	Find tan. at vertex by § 1 (7).
(3) P, tan., S.	Draw circle through S, P touching tan. in T. Use § 5 (2). (Two solutions.)
(4) P, Q, axis.	Use § 3 (1).
(5) 2 tans., axis.	$\angle STQ$ (Fig. 20) = angle which TP makes with axis.
(6) P, tan., axis.	Find image of P in axis. Use § 6 (3) to find pt. of contact of tan.
(7) P, Q, R, direction of axis.	Draw $QE \parallel$ axis meeting PR in E. Produce EQ, making $TQ : EQ = PE : ER$. By § 4 (8) T is on tan. at P. Similarly, or by § 4 (2), find tan. at Q. Then focus.
(8) 3 tans., direction of axis.	Use § 5 (2) or § 7 (2).
(9) P, Q, tan., direction of axis.	Use § 6 (3) to find p, pt. of contact of the tan. Bisect PQ in V. Draw $VV' \parallel$ axis meeting tan. in t, and $pV' \parallel PQ$. The m. pt. tV' is on curve. Draw tan. at this pt. Find S. (Two solutions.)
(10) P, 2 tans., direction of axis.	Determine direction of S as in (5) above. From T, intersection of tans., draw $TV \parallel$ axis. Draw any line between tans. bisected by TV. (*Use*: diagonals of $\parallel^m$ bisect each other.) Draw $PV \parallel$ this line. Produce $PV = VP'$. P' is on curve. Produce $P'P$ to meet one of tans. and use § 6 (3) to determine pt. of contact, &c.
(11) P, Q, R, S.	Let PQ, RS meet in O. Find $OF^2 = OP \cdot OQ$, $OE^2 = OR \cdot OS$, F, E in PQ, RS, $\therefore$ EF is direction of axis [§ 6 (3)]. (Two solutions.)
(12) 4 tans.	Use § 7 (1) to find S.
(13) P, Q, R, tan.	Use § 6 (3).
(14) P, Q, 2 tans.	Require Involution. [See § 34.]
(15) P, 3 tans.	Require Involution. [See § 34.]

12. Locus a Parabola. The following are the propositions used to prove the locus of a moving point to be a parabola:

§ 1 (2). $SP = PM$.

§ 1 (4). $PN^2 = 4AS \cdot AN$.

§ 4 (7). $QV^2 = 4AS \cdot PV$.

§ 4 (9). If fixed points QQ' can be found and PN be drawn in a fixed direction to meet QQ' in N so that PN varies as $QN \cdot NQ'$, the locus of P is a parabola.

§ 4 (8). OA, OB are fixed lines, and A a fixed point, MN in a fixed direction (M on A, N on B), P is taken on MN such that $NP : PM = OM : MA$. Locus of P is a parabola.

13. Envelope a Parabola. The following are the chief propositions used in proving the envelope of a moving line to be a parabola:

§ 1 (7). If moving line meets a fixed line in Y and perpendicular to it through Y passes through a fixed point.

§ 5 (2). If moving line PT meets fixed line QT in T and there exists any point S such that $\angle STP$ is const.

§ 7 (1). Two fixed lines intersect in O, S is a fixed point, the chord cut off by the fixed lines from any circle through O, S envelops a parabola.

§ 7 (4). If a variable line cuts three fixed lines in L, M, N and $LM : MN = \text{const.}$

§ 7 (3). If a variable line pr moves between two fixed lines qR, qp [Fig. 23] so that $Rp : pq = qr : rP$.

§ 7 (3). If a variable line pr cuts off intercepts x, y from two fixed lines (qR, qP) such that $\lambda x + \mu y = \text{const.}$, where λ and μ are const. This follows easily from § 7 (3).

Note.—In all envelope problems efforts should be made to gain a preliminary knowledge of the curve by taking limiting positions of the moving line. This should also be done in locus problems.

ELLIPSE.

14. Linear Properties, Principal Diameters.

(1) $CA = a$, $CB = b$, $e^2 = 1 - b^2/a^2$;
$CS = ae$, $CX = a/e$, $CS \cdot CX = CA^2$; $SB = a$, $SL = b^2/a$.

(2) $SP = e \cdot PM$.

(3) (*Adams*) $SD = e \cdot RQ$.

(4) $PN^2 : AN \cdot A'N = BC^2 : AC^2$
$Pn^2 : Bn \cdot B'n = AC^2 : BC^2$ } $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$.

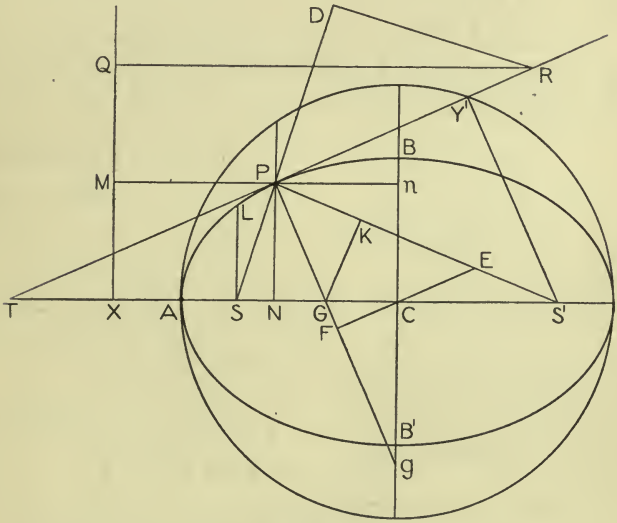


Fig. 26.

(5) $SP + S'P = 2a$, $SP = a - ex$, $S'P = a + ex$. The image of one focus in a tan. is dist. $2a$ from the other focus.

(6) $SY \cdot S'Y' = b^2$, and Y, Y' lie on the auxiliary circle.

(7) $CN \cdot CT = a^2$, $Cn \cdot Ct = b^2$; $CG \cdot CT = a^2 - b^2$.

(8) $CG = e^2 \cdot CN$. $\left[Cg = \frac{e^2}{1 - e^2} Cn = \frac{CS}{SX} \cdot Cn.\right]$

(9) $PF \cdot PG = b^2$, $PF \cdot Pg = a^2$.

(10) $SG = e \cdot SP$, $S'G = e \cdot S'P$.

(11) Projection of PG on either focal distance $(PK) = SL$.

" " Pg " " " " $= a$.

$\angle PEg = 90°$, $PE = a$.

(12) Tangents at P, P' intersect on axis.

(13) $PN : P'N = b : a$.

(14) Semi-latus rectum is a **H. M.** between the segments of any focal chord.

(15) *Frégier's Theorem.* If P is a fixed point, MN a variable chord such that $\angle MPN$ = rt. $\angle$, MN passes through a fixed point (**Frégier Point**) on the normal at P.

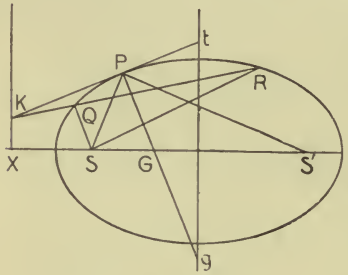


Fig. 27.

15. Angle Properties.

(1) KS bisects ext. $\angle RSQ$.

(2) $\angle KSP = 90°$.

(3) Tangents at the end of a focal chord intersect on the directrix.

(4) The joins of ends of two focal chords intersect on the directrix.

(5) PG, PK are internal and external bisectors of SPS'.

(6) $SgS'tP$ is cyclic.

16. Two Tangents.

(1) $\angle TSP = \angle TSQ$, $\angle TS'P = \angle TS'Q$. [Fig. 28.]

(2) $\angle STP = \angle S'TQ$.

(3) Tangents at rt. angles intersect on the **director circle.** rad. $\sqrt{a^2 + b^2}$.

(4) $TP:TQ =$ diam. $\parallel TP$: diam. $\parallel TQ$.

(5) If PQ meets S-directrix in K, $\angle TSK =$ rt. $\angle$.

(6) CT bisects PQ and all chords $\parallel PQ$.

(7) If p is any point on the arc PQ, and pL, pM, pN are drawn in const. directions to meet PQ, TP, TQ, pL^2 varies as $pM \cdot pN$ $[\alpha^2 = k\beta\gamma]$.

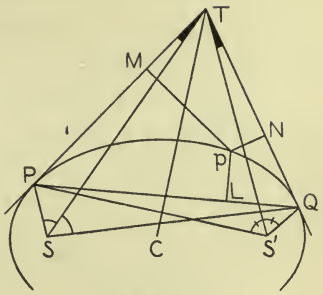


Fig. 28.

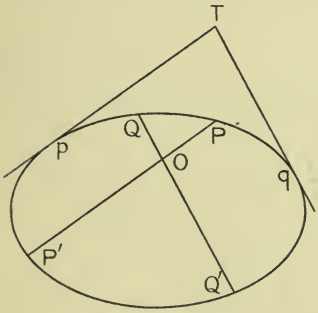


Fig. 29.

17. Intersecting Chords.

(1) If the directions of PP', QQ' are fixed,

$(PO \cdot OP')/(QO \cdot OQ')$

is independent of O,

$= Tp^2/Tq^2$,

= ratio of sqs. of $\parallel$ diams.

= ratio of $\parallel$ focal chords.

(2) If a circle intersects a conic in four points A, B, C, D, then AB, CD are equally inclined to the axis.

18. Three Tangents.

(1) The segment of a variable tangent between two fixed tangents subtends a constant angle at S and also at S', i.e. $\angle MSN$ is const., if RP, RQ are fixed, MN variable.

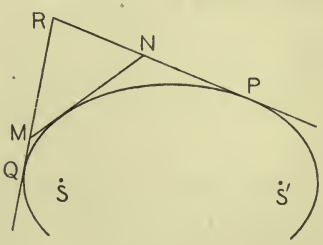


Fig. 30.

Three Tangents, two parallel.

(2) CR, CR' are conjugate, as are diagonals of any escribed parallelogram.

(3) $\Delta\, SRP \,|||\, \Delta\, SR'P'$,
$\Delta\, S'RP \,|||\, \Delta\, S'R'P'$.

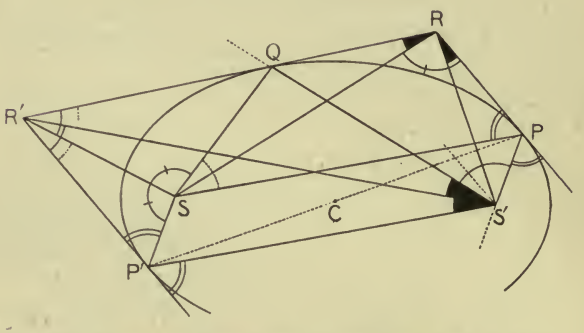


FIG. 31. The dotted lines in the figure indicate the lines necessary for proof.

(4) $\angle\, RSR' = \frac{1}{2}$ re-entrant $\angle\, PSP'$,
$\angle\, RS'R' = \frac{1}{2}\, \angle\, PS'P'$.

(5) $RP : R'P' = RQ : R'Q'$.

(6) $RP \,.\, R'P' = SP \,.\, S'P = SP' \,.\, S'P'$.

(7) The figure indicates most of the groups of equal angles which occur.

19. CONJUGATE DIAMETERS.

(1) $CD \parallel$ tan. at P and bisects chords $\parallel$ to it,
CP ,, ,, D ,, ,, ,, ,,

(2) Supplemental chords $\parallel$ conj. diams.

(3) $\angle\, P'CD' = 90°$.

(4) $CM = P'N$, $CN = D'M$.

(5) $CP^2 + CD^2 = a^2 + b^2$.

(6) Area $\Delta\ CI'D$
$= \text{const.} = \frac{1}{2}ab$,
i.e. $PF \,.\, CD = ab$,
i.e. $CD = ab/p$.

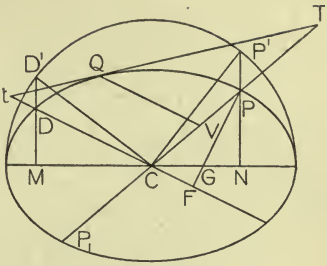


Fig. 32.

(7) $PG = \frac{b}{a}CD$;

$Pg = \frac{a}{b}CD$.

(8) $SP \,.\, S'P = CD^2$.

(9) $CV \,.\, CT = CP^2$
$[QV \,.\, Ct = CD^2]$.

(10) $QV^2/PV \,.\, VP_1 = CD^2/CP^2$, i.e. $x^2/a^2 + y^2/b^2 = 1$.

(11) $TQ \,.\, Qt =$ sq. of semidiam. conj. to CQ,
$= SQ \,.\, S'Q$.

20. Construction of Tangent [S = focus, d = directrix].

i. *P on curve.* (1) Given S, d. Use § 15 (2).

(2) Given S, S'. Use § 15 (5).

(3) Curve only. Pascal's Theorem, § 33.

ii. *P outside curve.* (1) Given S, d. Use § 14 (3).

(2) Given S, S', $2a$. Circle cent. S, rad. $2a$, and $\odot$ cent. P, rad. PS', cut in K, L. Join SK, SL meeting curve in T, t, then PT, Pt are tangents.

(3) Given S, S', $2a$. Circle on SP as diameter cuts the auxiliary $\odot$ in Y, Y'; PY, PY' are tangents.

(4) Join CP meeting curve in p, take V such that $CV \,.\, CP = Cp^2$. Through V draw ordinates QVQ'; PQ, PQ' are tangents.

21. Construction of Curve. (See also § 34.) The data here may be indefinitely varied between C (2 pts.), S (2), position of axis ($ax = 2$), length of axis ($a = 1$), direction of axis (1), directrix (d, 2), eccentricity (e, 1), asymptotes (4), direction of one asympt. (1), one asympt. (2), &c., and one or more up to five points (P, P,...) or tangents (t, t,...).

The following cases are arranged in classes, according to order of difficulty:

i. *Sde*; *SdP*; *Sdt*; *SS'P*; *SS't*; *SePP*; *S, e, ax, P*; *SPPP*; *Sttt*; *SaPP*; *SaPt*; *Satt*; *dPPP*; *dePP*.

ii. *C, e, ax, P*: Find *G, T, A* in this order.
C, e, ax, t: *PN/NG* is known, ∴ *PN/CN*, ∴ *P*.
C, ax, PP: Use § 14 (4) or Pascal's Theorem [§ 33].

iii. *C, P, Q, R*: Determine three other pts. and use Pascal. *Or*, Bisect *PQ* in *M*. *CB* ∥ *PQ* and *CM* are two conj. diams. *RR'* ∥ *CM*. Hence *R'* on curve. Let *RR'*, *PQ* meet in *E*. Then
$EP \cdot EQ : ER \cdot ER' = CB^2 : CA^2$ [§ 17 (1) *CA* = semidiam. along *CM*],
$= RN^2 : CA^2 - CN^2$ [§ 19 (10) *N* is pt. where *RN* ∥ *CB* meets *CM*].
Hence *CA, CB*, &c.

The following are notes on the more important cases:

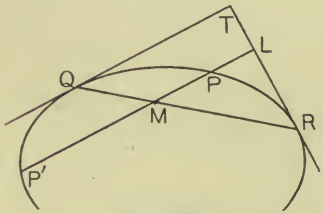


FIG. 33.

iv. *PPPPP*. Use § 17 (1) *or* see § 34.

v. *P, tt and their points of contact*. Given *P, TQ, TR, Q, R* (Fig. 33).

$$LP \cdot LP'/LR^2 = TQ^2/TR^2 = LM^2/LR^2;$$

∴ $LP \cdot LP' = LM^2$, ∴ *P'* is known, ∴ diam. through *Q*.

vi. *Two Conjugate Diameters CP, CD*.

Through *P* draw ⊥r *CD* and on it take $PK = PK' = CD$. Axes bisect ∠ *KCK'*, trv. axis in acute ∠ *KCK'*. $CK' = a+b$, $CK = a-b$. *Or*, Circle *KCK'* meets ∥ through *P* to *CD* in *T, t*. *CT, Ct* are directions of axes. Then use § 14 (7).

vii. *PPPPP, PPPPt, PPPtt, PPttt, Ptttt, ttttt* can all be solved by Theory of Polars and Harmonics. See § 34.

22. ELLIPTIC LOCI. The following are the most important propositions used in proving loci to be ellipses:

(1) $SP = e \cdot PM$.

(2) $SP + S'P = \text{const.}$

(3) $PN^2/(AN \cdot A'N) = \text{const.}$

(4) $QV^2/(PV \cdot VP_1) = \text{const.}$

(5) § 16 (7), i.e. if P moves so that if α, β, γ be its perpendiculars upon the sides of a triangle and $\alpha^2 = k\beta\gamma$, then P describes a conic touching AB, AC at B and C.

(6) If the perpendiculars on four fixed lines be such that $p_1 p_2 / p_3 p_4 = \text{const.}$ $[\alpha/\beta = k \,.\, \gamma/\delta]$. [*Milne and Davis*, p. 201.]

(7) OP, OQ are two fixed lines, PQ a const. length, any point on PQ describes an ellipse.

(8) More generally: OP, OQ are two fixed lines, PQR is a given Δ, R describes an ellipse. [*Macaulay*, p. 75.]

23. Elliptic Envelopes. The following are the most important propositions employed in proving envelopes of lines to be ellipses, &c.:

(1) If the foot of the $\perp^r$ on the line from a fixed point lies on a fixed circle. § 14 (6).

(2) If there be two fixed points, S and S', the product of the perpendiculars from which is const. § 14 (6).

(3) If X, Y move on two fixed lines OX, OY and

(*a*) $\angle XSY = \text{const.}$, S being a fixed point, the envelope of XY is a conic. [§ 18 (1).]

(*b*) Area $\Delta XOY = \text{const.}$, the envelope is a hyperbola with OX, OY asymptotes.

(*c*) $\lambda \,.\, OX + \mu \,.\, OY = \text{const.}$, the envelope is a parabola. [See §§ 7 (3), 13.]

24. Curvature.

(1) The central chord of curvature $= 2CD^2/CP$.

(2) The focal chord of curvature $= 2CD^2/CA$.

(3) The normal chord of curvature $(2\rho) = 2CD^2/p$.

(4) Thus $\rho = CD^2/p = CD^3/ab = PG^3/SL^2 = a^2b^2/p^3$.

(5) *To draw* circle of curvature at P: Let normal meet axis in G. Draw $GK \perp^r PG$ meeting a focal distance in K. At K draw $\perp^r$ to this focal distance which will meet normal in centre of curvature.

HYPERBOLA

[N.B.—*To economize space, properties given for the ellipse are not usually repeated here. Angles which are equal in the ellipse are, in the hyperbola, either equal or supplementary.*]

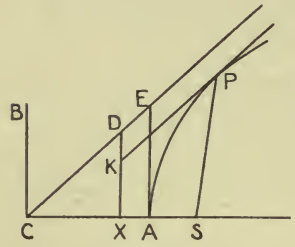


Fig. 34.

25. Metrical Properties.

(1) $AE = SD = BC$.

(2) $CD = CA$, i.e. D is on the auxiliary ⊙ to which SD is a tangent.

(3) $CE = CS$.

(4) If PK is parallel to an asymptote, $SP = PK$.

(5) $S'P - SP = AA'$.

(Fig. 38.)

(6) $PR = P'R'$,
$Pr = p'r'$,
$PT = PT'$.

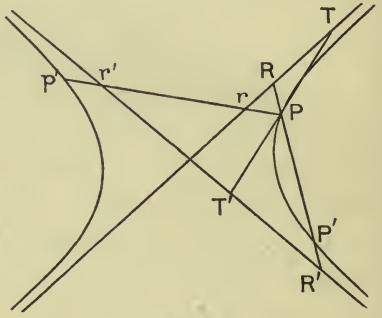


Fig. 35.

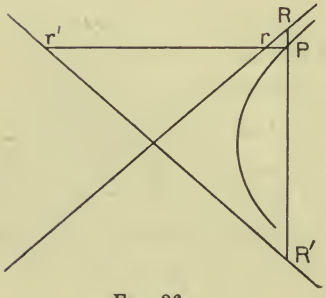


Fig. 36.

(7) $PR \cdot PR' = BC^2$
$Pr \cdot Pr' = AC^2$,

if RPR' and Prr' are parallel to the axes.

(8) $RQ \cdot QR' = $ const. for $\parallel$ chords,
$= PT'^2 = CD^2$.
So $rQ \cdot Qr' = CP^2$ if
$rQ \parallel CP$.

(9) $\Delta TCT'$
$=$ const. area $= AC \cdot BC$,
i.e. $CT \cdot CT' = $ const.
$= CS^2$,
i.e. $PH \cdot PK = $ const.
$= \frac{1}{4} CS^2$.

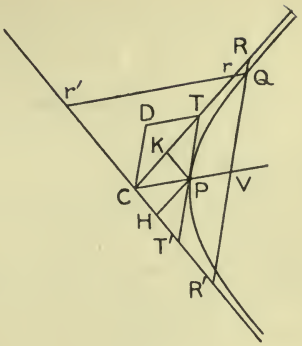


Fig. 37.

(10) One asymptote is the diagonal of the $\parallel^m$ completed out of CP, CD, and PD is $\parallel$ to the other. This parallelogram has a constant area ($AC \cdot BC$).

(11) To describe a hyperbola, given CP, CD, use (9) and (8). See also § 21 (vi).

26. Angle Properties.

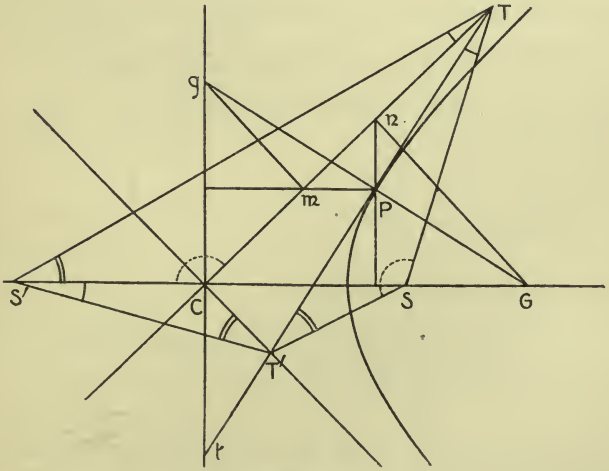


Fig. 38.

(1) **Cyclic Groups** (Fig. 38):

(α) $STS'T'$. (β) $CTT'Gg$. (γ) $SPgS't$.

(2) **Similar Triangles** (Fig. 38):

(α) $\triangle TST' \mid\mid\mid \triangle TCS' \mid\mid\mid \triangle T'CS'$, and the marked angles are equal.

(β) $\triangle TS'T' \mid\mid\mid \triangle TCS \mid\mid\mid \triangle T'CS$.

(3) $\angle CnG$ = rt. $\angle = \angle Cmg$. Hence a good method of drawing the normal and therefore the tangent at any point on the curve.

RECTANGULAR HYPERBOLA

27. Metrical Properties.

(1) The eccentricity = $\sqrt{2}$.

(2) Diameters which make equal angles with either axis are equal.

(3) Diameters which are conjugate are equal.

(4) Diameters which are at rt. angles are equal.

(5) $PG = CD = Pg$, i.e. P is the centre of the $\odot CTT'Gg$. [See § 26 (1) (β).]

(6) $PN^2 = AN \cdot A'N$,

$QY^2 = PV \cdot VP'$.

28. Angle Properties.

(1) Conjugate diameters make $\left\{\begin{matrix}\text{equal}\\ \text{complementary}\end{matrix}\right\}$ angles with either $\left\{\begin{matrix}\text{asymptote}\\ \text{axis}\end{matrix}\right\}$.

(2) The angle between any two diameters = $\angle$ between their conjugates.

(3) $\angle QPT = \angle QpP$.

(4) Any chord of a R. H. subtends equal (or supplementary) angles at the end of any diameter.

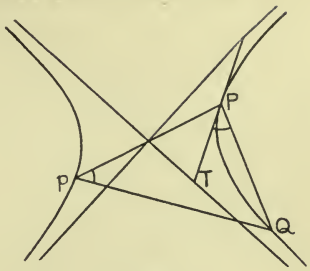


Fig. 39.

(5) If a R. H. circumscribe a Δ it passes through the orthocentre, and the locus of its centre is the N. P. C. of the Δ.

(6) All conics through the intersection of two R. H.'s are R. H.'s.

(7) If PQ, PR be two chords at rt. angles, QR is $\parallel$ to the normal at P.

Proof.—The R. H. through P, Q, R passes through orthocentre PQR, $\therefore$ $\perp^r$ from P on QR is tan. at P.

Note.—Hence in a R. H. the Frégier point of every pt. is at ∞.

29. Loci.

The following proposition is of service in proving loci to be R. H.'s:

If A and B are fixed points and P is such that

$$\angle PAB - \angle PBA = \text{const.},$$

the locus of P is a R. H. of which AB is a diameter.

Proof.—Use § 28 (4).

HARMONICS AND GENERAL THEOREMS

30. Pole and Polar.

(1) If the polar of P passes through Q, the polar of Q passes through P.

If the pole of p lies on q, the pole of q lies on p.

Conjugate Points: Points such that the polar of each passes through the other.

Conjugate Lines: Lines such that the pole of each lies on the other.

(2) If the polars of P and Q intersect in R, the polar of R is PQ.

If the poles of p and q lie on r, the pole of r is pq.

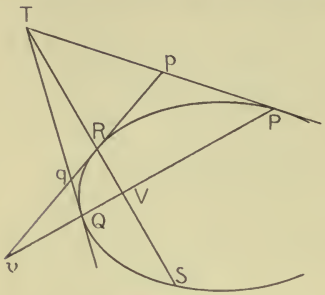


Fig. 40.

(3) The line joining any pair of conjugate points is cut harmonically by the conic. Particular Cases:

(a) $\{TRVS\} = -1$,

(b) $\{PVQv\} = -1$,

where v is pole of RS.

(4) $\{pRqv\} = -1$, i.e. if two tangents be drawn to a conic any third tangent is harmonically divided by the two tangents, the curve, and the chord of contact.

31. Circumscribed and Inscribed Quadrilaterals. [See *GEOM.* § 24.]

(1) The harmonic triangle of a 4^1 circsd. about a conic is self-conjugate [self-polar].

(2) The harmonic triangle of a 4^1 inscd. in a conic is self-conjugate [self-polar].

(3) The harmonic triangle of a 4^1 circsd. to a conic coincides with the harmonic triangle of the 4^1 formed by the points of contact. This includes the following propositions:

(a) The int. diagls. of an escd. 4^1 and of the corresponding inscd. meet in a pt.

(b) The third diagls. of these 4^1s are collinear.

(c) Any pair of opposite sides of an inscd. 4^1 intersect on a diagl. of the corresponding escd. 4^1.

(4) The locus of the centre of a conic inscd. in a given 4^1 is the '*diameter*' of the 4^1.

(5) If a 4^1 is inscd. in a conic, any st. line is cut in involution by the 3 prs. of

(6) If a 4^1 is circsd. about a conic, the lines connecting any pt. with the vertices form

opposite connectors and by the conic. (**Desargues' Theorem.** See *GEOM.* § 24 (7).)

a pencil in involution and the pr. of tans. from the pt. also belong to the pencil.

Note.—By making two vertices of the 4^1 coincide we obtain the results of (7), (8). By making two pairs of opposite vertices coincide we obtain the results of (9), (10).

(7) If a Δ is insd. in a conic, any st. line is cut in involution by the conic, the two sides which meet at any vertex, the tan. at that vertex, and the opposite side.

(8) If a conic is insd. in a Δ, the two tans. from any ext. pt., the lines joining that pt. to the ends of any side, to the vertex opp. that side, and to its pt. of contact form a pencil in involution.

(9) If TP, TQ are tans. to a conic and a trv. meet TP, TQ in R, S, the conic in M, N, and PQ in F, then F is a double pt. of the involution (MN, RS).

(10) If TP, TQ are tans. to a conic and O be any pt., OT is a double line of the involution (OP, OQ, tans. from O).

32. Anharmonics and Homography. [See *GEOM.* § 31, 32 and *A. GEOM.* § 66.]

(1) If A, B, C, D are fixed pts. on a conic and P a variable point on it,

$$\{P.ABCD\} = \text{const.}$$

(2) If a, b, c, d are fixed tans. to a conic and p a variable tan.,

$$\{pa, pb, pc, pd\} = \text{const.}$$

(3) *Homographic Ranges on Conic.*

(*Def.*) $\{ABC\ldots\} = \{A'B'C'\ldots\}$,

the pts. all lying on conic if

$\{P.ABC\ldots\} = \{Q.A'B'C'\ldots\}$,

P and Q being any two pts. on conic.

(4) *Homographic Tangents to Conic.*

(*Def.*) $\{abc\ldots\} = \{a'b'c'\ldots\}$,

the lines all touching conic if

$\{p.abc\ldots\} = \{q.a'b'c'\ldots\}$,

p and q being any two tans. to conic.

(5) If $\{ABC\ldots\}$
$= \{A'B'C'\ldots\}$,
the pts. all lying on conic, then the points
$(AB', A'B)\ (AC', A'C)\ldots$
all lie on st. line.

(Homographic Axis.)

(7) If through pt. V outside a conic a pencil of rays is drawn cutting it in AA', BB',... then
$\{ABC\ldots\} = \{A'B'C'\ldots\}$,
the homographic axis being polar of P.

(6) If $\{abc\ldots\}$
$= \{a'b'c'\ldots\}$,
the lines all touching conic, then the lines
$(ab', a'b)\ (ac', a'c)\ldots$
all pass through a pt.

(Homographic Pole.)

(8) If from pts. on a line p outside a conic the pairs of tangents $(aa')\ (bb')\ldots$ are drawn to the conic, then
$\{abc\ldots\} = \{a'b'c'\ldots\}$,
the homographic pole being pole of v.

33. Some General Theorems.

(1) *Pascal's Theorem.* If A, B, C, D, E, F, be 6 pts. on conic, the pts. $(AB, DE)\ (BC, EF)\ (CD, FA)$ lie on st. line.

(2) *Brianchon's Theorem.* If a, b, c, d, e, f be 6 tans. to conic, the lines $(ab, de)\ (bc, ef)\ (cd, fa)$ pass through pt.

Notes.—(1) See § 36 (2) (f); also *GEOM.* § 31 (8), *A. GEOM.* § 65 (2).

(2) There are 60 possible *Pascal Lines* which intersect 3 by 3 in 20 *Steiner Points*, which lie 4 by 4 on 15 st. lines. Similarly for Brianchon's Theorem.

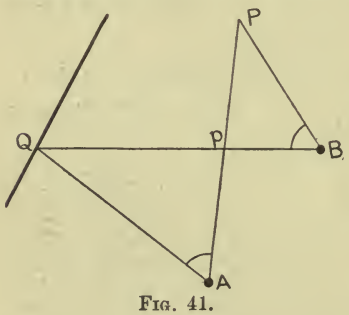


Fig. 41.

(3) *Newton's Theorem.*

If A and B be fixed points and Q move along any st. line and $\angle PAQ$ = const., $\angle PBQ$ = const., the locus of P is a conic through A and B.

Cor. When $\angle PBQ = 0$ the theorem gives that locus of p is a conic through A, B.

Proof.—$\{B.P\} = \{B.Q\}$ ($\because \angle PBQ$ is const.)
$= \{Q\} = \{A.Q\} = \{A.P\}$, &c.

(4) *Maclaurin's Theorem.*

The locus of the third vertex (P) of a Δ, two of whose vertices (Q, R) move along two fixed lines (OQ, OR) and whose sides pass through fixed pts. (A, B, C), is a conic passing through O, B, C.

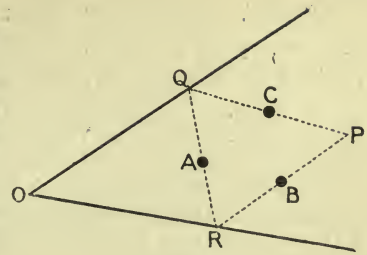


FIG. 42.

Note.—Taking Q, R at O it is clear O is on locus; so also pts. $(AC, OR)(AB, OQ)$.

Proof.—$\{B.P\} = \{B.R\} = \{R\} = \{A.R\} = \{A.Q\} = \{Q\} = \{C.Q\} = \{C.P\}$, &c.

34. CONSTRUCTION OF CONIC. (See also §§ 11, 21, and for notation used § 21.)

(1) *PPPPP.* [Particular Cases: *PPPt* and its pt. cont.; *PPPP* and dir. asympt.; *PPP* and one asympt.; *P* and both asympts., &c.]

Methods:

(*a*) The chord through any one pt. ∥ to chord of two other pts. can be easily determined by § 17 (1). Thence centre.

(*b*) Use Pascal to determine tans. at P. Thence centre.

(*c*) *By Newton's Theorem*: A, B, C, D, E the pts.; take ABC, BAC as the two fixed angles of Newton's Theorem, which generate conic by rotation round B, C as poles. Determine fixed line by pts. D, E.

(*d*) *By Maclaurin's Theorem* (*Note*). The five given pts. are, in the fig., O, (AC, OR), (AB, OQ), B, C. Hence A. Hence Δ, &c.

(*e*) If AC, AD, AE, BC, BD, BE meet *any* line in C_1, D_1, E_1, C_2, D_2, E_2, the common pts. of the ranges $\{C_1D_1E_1\}$, $\{C_2D_2E_2\}$ are on conic. Draw the line through $C \parallel BD$. Hence centre.

(*f*) Use § 31 (2). Take any four pts. $ABCD$. Let $F = (AC, BD)$, $G = (AD, BC)$, $H = (AB, CD)$. Let E be fifth pt. Join B to (AE, GF). If this join meet EH in E' then E' is on conic, &c.

(2) *ttttt.* [Particular Case: § 11 (12).] Reciprocation may be applied to (1), (*b*), (*d*), (*e*), (*f*). The readiest solution is given by Brianchon's Theorem.

(3) *PPPPt.* [Particular Case: § 11 (11).] Use § 31 (5). The four pts. give the 4^1, *t* the line. The pt. of contact of *t* is found as one of the double points of the involution. Case now reduces to *PPPP*.

(4) *tttP.* [Particular Case: § 11 (15).] Reciprocal of last case.

(5) *PPPtt.* [Particular Case: § 11 (13).] *tttPP.* [Particular Case: § 11 (14).] Use § 31 (9) and § 31 (10) respectively.

35. RECIPROCATION.

(1) With respect to any conic:

Reciprocal Terms.

Point; collinear; line; locus; inscribed.	Line; concurrent; point; envelope; circumscribed.
Join of two points.	Intersection of two lines.
Point on conic.	Tangent to conic.
Conics with $\left\{\begin{matrix}\text{single}\\\text{double}\end{matrix}\right\}$ contact.	Conics with $\left\{\begin{matrix}\text{single}\\\text{double}\end{matrix}\right\}$ contact.
Chord of intersection of two conics.	Intersection of common tangents to two conics.
Pole.	Polar.
Line at infinity.	Centre of reciprocating conic.
Conic.	Conic: ellipse, parabola, or hyperbola as the tangents to the original conic from the centre of the reciprocating conic are imaginary, coincident, real.

(2) *With respect to a Circle* ('*Point*', '*Origin*').

(*a*) The reciprocal of a *circle* whose centre is *C*, rad. *a*, with respect to a circle, centre *O*, radius *k*, is a conic with

Focus: *O*. *Eccentricity*: OC/a.

And therefore ellipse, parabola, or hyperbola as $OC < = \text{or} > a$.

Second Focus: Find centre and use (*c*) (α).

Semi-latus rectum: k^2/a.

Directrix: Reciprocal of C.

This is the directrix belonging to the O focus.

Centre: Reciprocal of polar of O w. r. circle C.

Direction of Axis: OC.

Note.—These results are so easily found, by polar coordinates, that it is questionable whether it is worth while to memorize them.

(*b*) *Concentric circles* reciprocate into conics with common focus and directrix.

Coaxal circles

(α) Reciprocate into confocal conics if O be a L.P.

Note.—The R.A. reciprocates into the other focus.

(β) Reciprocate into conics with one focus common and equal minor axes if O be on rad. axis.

Equal circles reciprocate into conics with one focus common and equal latera recta.

(*c*) *Magnitude Properties.* The only properties easily applicable are:

(α) The distance of any point P from origin varies inversely as the distance of the reciprocal line.

(β) The A.R. of a range = that of the reciprocal pencil.

(*d*) *Angle Properties.* The angle between any two lines is equal to the angle subtended at the origin by the line joining their reciprocals.

(*e*) *Reciprocal Terms.*

Centre of reciprocation, O.	Line at infinity.
Line through O.	Point at infinity.
Parallel lines.	Points in same st. line as centre of reciprocation.
Conic: O anywhere.	Conic, see § 35 (1), also note below.

Reciprocal Terms (continued).

Conic: O on director circle.	Rectangular hyperbola.
„ : O at focus.	Circle, see § 35 (2) (*a*).
„ : O at centre.	Similar conic.
Parabola: O on directrix.	Rectangular hyperbola.
Confocal conics: O at one of the foci.	Coaxal circles: O a limiting point (§ 35 (2) (*b*)).
Concentric conics.	Conics with respect to which O has the same polar.
Conics with same eccentricity.	Conics touching a pair of lines which meet at O.
Conics touching at O.	Parabolas with parallel axes.
Conjugate diameters.	Two points which, with the centre of reciprocation, form a self-conjugate triad with reciprocal conic.
Harmonic range.	Harmonic pencil.
Triangle, O at orthocentre.	Triangle, O at orthocentre.

Note.—If OP, OQ be the tangents from O to the conic which is to be reciprocated we have for the reciprocal conic:

Asymptotes: Reciprocals of P and Q.
Centre: Reciprocal of PQ.
Axes: Parallel to the two bisectors of $\angle POQ$.
Eccentricity: Depends on $\angle POQ$ only.

36. Projection.

(1) *Properties unaltered by Projection*:

Degree of curve.

Tangency of curves, or of lines and curves.

Relations of poles and polars.

Anharmonic ratio of ranges and pencils.

Any relation connecting the distances of points in a straight line, provided that every term of the relation mentions the same points, although in different orders.

Note.—Under certain limitations the points need not be in a st. line, e.g. Ceva's Theorem is projective.

(2) *Simplifications of Figure.*

(*a*) Any line may be sent to infinity and ANY TWO angles into given angles.

(*b*) Any quadrilateral into square. Send third diagonal to ∞. Project one $\angle$ and $\angle$ between diagonals into rt. angles.

(*c*) Any triangle into equilateral Δ. Project any two of its angles into $\pi/3$.

Note.—Any 4^1 may *simultaneously* be made into a $\|^m$, or any line projected to ∞. The Δ may be made of given species.

(*d*) Conic into parabola. Send any tangent to ∞.

Note.—Any triangle may *simultaneously* be made equilateral.

(*e*) Conic into circle with projection of given point as centre. Send polar of point to ∞ and any two angles in segment of conic cut off by a line through the point into rt. angles.

(*f*) Conic into circle and any line to ∞. Take the pole of the line. Draw any two angles with vertices on conic and standing on chords passing through the pole. Project these into rt. angles and the line to ∞.

Note.—Hence an easy proof of Pascal's Theorem.

(*g*) Conics through two given points into circles. Project line joining points to ∞ and one of the conics into a circle by (*f*).

(*h*) Conics through four given points into coaxal circles.

(*i*) Conics touching at two given points into concentric circles. As in (*g*).

(*j*) Conic into hyperbola of given eccentricity with Send chord of contact of tangents through C to ∞ and

centre at projection of given point C.

project $\angle$ between them into the given angle between the asymptotes.

Note.—Any pair of lines may *simultaneously* be projected into $\parallel$ lines, or any angle into a given angle.

(k) Conic into conic with projections of given points C and S for centre and focus.

Let RX be polar of S, X on CS, RP tan. to conic. S end polar of C to ∞ and project PSR, SXR into rt. angles.

(3) *Focal Properties.* To project these we must use the definition of a focus given in *A. GEOM.* § 55 Note (2).

Ex. Confocal conics project into conics inscribed in the same quadrilateral.

(4) *Angle Properties.* These are projected by the theorem: The pencil formed by the two legs of a given angle and the imaginary lines through its vertex to the circular points at ∞ (focoids) has a given anharmonic ratio.

Ex. Lines at rt. angles project into lines which cut harmonically the line joining the two fixed points which are the projections of the focoids.

(5) *The Casey Projection.* (Casey's *A. Geom.*, pp. 271–7.) Casey reduces the treatment of this subject to two dimensions.

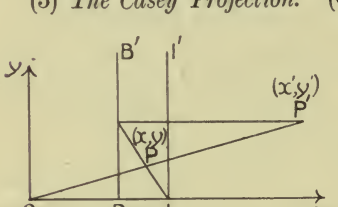


Fig. 43.

Ox, Oy are the axes.

BB', II' are two lines $\parallel$ to Oy cutting Ox in B and I.

The '**projection of** P' is found by joining OP, IP, the latter meeting BB' in C and drawing $CP' \parallel$ to Ox, meeting OP in P'.

Infinity Line. Any point on II' is projected to ∞.

Coordinates. $x = \dfrac{ax'}{c+x'}$, $y = \dfrac{ay'}{c+x'}$.

The properties of projections can all be readily proved by this construction, which gives results identical with those of the older method.

ANALYTICAL GEOMETRY

RELATIONS OF POINTS

1. *Distance* $(x_1\,y_1)\,(x_2\,y_2)$ is

$$\delta = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+2\,(x_1-x_2)\,(y_1-y_2)\cos\omega}.$$

2. (1) The coordinates of the points dividing $(x_1\,y_1)\,(x_2\,y_2)$ in the ratio $n_1 : n_2$ are :

$$x = \frac{n_2\,x_1+n_1\,x_2}{n_2+n_1} \qquad x = \frac{n_2\,x_1-n_1\,x_2}{n_2-n_1}$$

(Internal) and (External).

$$y = \frac{n_2\,y_1+n_1\,y_2}{n_2+n_1} \qquad y = \frac{n_2\,y_1-n_1\,y_2}{n_2-n_1}$$

(2) The coordinates of mid-point of $(x_1\,y_1)\,(x_2\,y_2)$ are

$$x = \tfrac{1}{2}\,(x_1+x_2),\quad y = \tfrac{1}{2}\,(y_1+y_2).$$

(3) The coordinates of centroid of $(x_1\,y_1)\,(x_2\,y_2)\,(x_3\,y_3)$ are

$$x = \tfrac{1}{3}\,(x_1+x_2+x_3),\quad y = \tfrac{1}{3}\,(y_1+y_2+y_3).$$

3. The *area* of the $\Delta\,(x_1\,y_1)\,(x_2\,y_2)\,(x_3\,y_3)$ is

$$\Delta = \pm\tfrac{1}{2}\,\{x_1\,(y_2-y_3)+x_2\,(y_3-y_1)+x_3\,(y_1-y_2)\}\sin\omega,$$

$$= \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\sin\omega.$$

Proof.—Find $\Delta\,OAB = \tfrac{1}{2}\,(x_2\,y_1-x_1\,y_2)\sin\omega$,

$$\Delta\,ABC = OBC-OCA-OAB.$$

This method applies readily to polygons.

Notes.—(1) For rectangular axes put $\omega = 90°$, $\cos\omega = 0$, $\sin\omega = 1$.

(2) The *sign* of the area depends on whether a man walking round it from $(x_1\,y_1)$ to $(x_2\,y_2)\,(x_3\,y_3)$ keeps it on his left (sign + in above formula) or right (−). This is important when two areas are added.

4. Polar Coordinates.

$$x = r\cos\theta, \qquad r^2 = x^2 + y^2,$$
$$y = r\sin\theta, \quad \tan\theta = y/x.$$

Notes.—(1) The point (r, θ) may also be described as $(-r, \theta+\pi)$, &c.

(2) But of these descriptions one may satisfy an equation while the other does not, e.g. the point $(2, 0)$ satisfies $r = 1+\cos\theta$, the point $(-2, \pi)$ does not.

5. $\delta^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$,
$\Delta = \mp \frac{1}{2} \Sigma r_2 r_3 \sin(\theta_2 - \theta_3)$.

THE STRAIGHT LINE

6. Forms of Equation ($\omega = 90°$).

(1) $y = mx + c$. m is the tangent of the angle which the part of the line *above* the x-axis makes with the + direction of that axis, c is the intercept on the y-axis.

(2) $\frac{x}{a} + \frac{y}{b} = 1$. a, b are intercepts on axes. This equation holds for oblique axes.

(3) $\left.\begin{aligned} Ax + By + C &= 0 \\ rx + sy &= 1 \end{aligned}\right\}$. Generalized forms of the above.

(4) $x\cos\alpha + y\sin\alpha = p$. (p, α) are polar coordinates of foot of $\perp^r$ from O on line. To put any equation in this form: *Divide throughout by the square root of the sum of the squares of the coefficients of x and y.*

(5) $\left.\begin{aligned} x &= h + r\cos\theta \\ y &= k + r\sin\theta \end{aligned}\right\}$. The two *together* constitute the equation, (h, k) is any point on the line, and (r, θ) the polar coordinates of the current point (x, y) with reference to (h, k). [See also §§ 11, 13.]

7. Inclination of Straight Lines.

$y = m_1 x + c_1,$	$x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0,$
$y = m_2 x + c_2.$	$x\cos\alpha_2 + y\sin\alpha_2 - p_2 = 0.$
(1) $\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1 m_2}.$	$\theta = \pm(\alpha_1 - \alpha_2).$

The lines are parallel if

(2) $m_1 = m_2$. $\alpha_1 - \alpha_2 =$ zero or a multiple of π.

The lines are perpendicular if

(3) $1 + m_1 m_2 = 0$. $\alpha_1 - \alpha_2 =$ odd multiple of $\frac{\pi}{2}$.

Note.—For the lines $A_1 x + B_1 y + C_1 = 0$, $A_2 x + B_2 y + C_2 = 0$, the conditions of parallelism and perpendicularity are $A_1 B_2 - A_2 B_1 = 0$, $A_1 A_2 + B_1 B_2 = 0$.

8. Lines under Conditions.

(1) Any line through the origin: $y = mx$.

,, ,, ,, (x_1, y_1): $y - y_1 = m(x - x_1)$.

(2) The join of (x_1, y_1) (x_2, y_2): $y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$,

or, $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

(3) Any line through the intersection of $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ is of form

$$A_1 x + B_1 y + C_1 + \lambda (A_2 x + B_2 y + C_2) = 0.$$

(4) The line through (x_1, y_1) $\parallel$ to $y = mx + c$ is

$$y - y_1 = m(x - x_1).$$

The line through (x_1, y_1) $\perp^r$ to $y = mx + c$ is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

The line through (x_1, y_1) $\perp^r$ to $x \cos\alpha + y \sin\alpha = p$ is

$$y - y_1 = \tan\alpha\,(x - x_1).$$

(5) All lines $\parallel$ to $Ax + By + C = 0$ are included in $Ax + By + D = 0$, all lines $\perp^r$ to it in $Bx - Ay + D = 0$.

Thus the line through (x_1, y_1) $\perp^r$ to it is

$$Bx - Ay = Bx_1 - Ay_1.$$

(6) The lines (A_1, B_1, C_1) (A_2, B_2, C_2) (A_3, B_3, C_3) concur if the determinant formed by the coefficients vanishes.

9. Length of Perpendicular.

The length of the perpendicular from (x, y) on $x\cos\alpha + y\sin\alpha - p = 0$ is $\pm (x\cos\alpha + y\sin\alpha - p)$.

The length of the perpendicular from (x, y) on

$$Ax + By + C = 0 \text{ is } \pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}}.$$

Rule.—Bring all terms to the same side, substitute coordinates of point, and divide by sq. root of sum of sqs. of coeffts. of x and y.

Note.—All points on *same side* of a line have perpendiculars of *same sign*. Hence we can determine if a point lies on same side as origin.

10. Bisectors of Angles between lines (p_1, α_1) (p_2, α_2) are

$$x\cos\alpha_1 + y\sin\alpha_1 - p_1 = \pm (x\cos\alpha_2 + y\sin\alpha_2 - p_2).$$

For other lines bring to (p, α) form first [§ 6 (4)]. This leads to the form $\frac{Ax+By+C}{\sqrt{A^2+B^2}} = \pm \frac{A'x+B'y+C'}{\sqrt{A'^2+B'^2}}$. To settle the sign: The sign for the bisector of the angle in which any pt. (h, k) lies is the sign of $(Ah+Bk+C)/(A'h+B'k+C')$.

11. Polar Equations.

(1) St. line through pole, $\theta = \alpha$.

(2) Any st. line, $p = r\cos(\theta - \alpha)$.

(p, α) have the same meaning as before [§ 6 (4)].

(3) The lines perpendicular to the st. line

$$k/r = A\cos(\theta - \alpha) + B\cos(\theta - \beta)$$

are included in $\frac{l}{r} = A\cos(\theta + \frac{\pi}{2} - \alpha) + B\cos(\theta + \frac{\pi}{2} - \beta)$.

12. Two Straight Lines.

(1) $ax^2 + 2hxy + by^2 = 0$ represents two straight lines through O which are:

(α) Coincident if $h^2 - ab = 0$.

(β) At right angles if $a + b = 0$.

(γ) At an angle θ given by $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$.

(2) The lines bisecting the angles between them are

$$\frac{x^2-y^2}{a-b}=\frac{2xy}{2h}.$$

(3) $ax^2+2hxy+by^2+2gx+2fy+c=0$,

i.e. $(a, b, c, f, g, h)(x, y, 1)^2=0$,

represents two st. lines if $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$

Proof.—For determinant form assume $(a, b, c, f, g, h)(x, y, z)^2$

$\equiv (px+qy+rz)(lx+my+nz)$.

Differentiate in x, in y, in z. In the three resulting equations choose (x, y, z) to represent the point of intersection of $px+qy+rz=0$, and $lx+my+nz=0$. Hence for this point $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+ez=0$, &c.

Notes.—(1) The determinant is easily remembered if the order in which the letters a, b, c, f, g, h occur in it is noted. In writing it down remember that f, g, h *occur with 2's* in the equation and are not coefficients.

(2) Its expanded form is $abc-af^2-bg^2-ch^2+2fgh=0$.

(3) The angle between the lines is the $\angle$ between $ax^2+2hxy+by^2=0$.

(4) The point where the lines intersect is $(G/C, F/C)$, where C, F, G are the minors of c, f, g in Δ. Hence by (2) the lines bisecting the angles between the lines.

13. Oblique Coordinates.

(1) The inclination of $y=mx+c$ to the x-axis is given by

$$\tan\theta=\frac{m\sin\omega}{1+m\cos\omega}.$$

(2) If the perpendicular p be inclined at angles α, β to the two axes a line may be written $x\cos\alpha+y\cos\beta=p$ To put $Ax+By+C=0$ in this form multiply by $\sin\omega$ and divide by $\sqrt{A^2+B^2-2AB\cos\omega}$.

(3) The angle between $y=m_1x+c_1$ and $y=m_2x+c_2$ is given by

$$\tan\theta=\pm\frac{(m_1-m_2)\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2}.$$

(4) These lines are parallel if $m_1=m_2$.

" " " perpendicular if

$$1+(m_1+m_2)\cos\omega+m_1m_2=0.$$

(5) The $\perp^r$ from (X, Y) on $Ax+By+C=0$ is

$$\pm \frac{(AX+BY+C)\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}}.$$

(6) The angle between $ax^2+2hxy+by^2=0$ is given by

$$\tan\theta = \frac{2\sin\omega\sqrt{h^2-ab}}{a+b-2h\cos\omega}.$$

14. Some General Principles.

(1) If $\alpha=0$, $\beta=0$ be st. lines, the line $\alpha+\lambda\beta=0$ (λ variable) always passes through the fixed point $\alpha=0$, $\beta=0$.

(2) If u_r stand for a homogeneous function of x and of degree r, the st. lines joining O to the intersections of

$$\left.\begin{aligned} u_r+u_{r-1}+\ldots+u_1+u_0 &= 0 \\ mx+ny &= 1 \end{aligned}\right\}$$

are $u_r+(mx+ny)\,u_{r-1}+(mx+ny)^2\,u_{r-2}+\ldots=0$.

Rule.—Render the curve-equation homogeneous by means of the line-equation.

(3) The equation to the lines joining O to the intersections of

$$\left.\begin{aligned} u_r+u_{r-1}+\ldots+u_1+u_0 &= 0 \\ v_r+v_{r-1}+\ldots+v_1+v_0 &= 0 \end{aligned}\right\},$$

is the λ-eliminant of

$$\left.\begin{aligned} u_r+\lambda\,u_{r-1}+\lambda^2\,u_{r-2}+\ldots &= 0 \\ v_r+\lambda\,v_{r-1}+\lambda^2\,v_{r-2}+\ldots &= 0 \end{aligned}\right\}.$$

Proof.—If P be a point of intersection and (x, y) any pt. on OP, then the coordinates of P may be written $(x/\lambda, y/\lambda)$. Express that this lies on both curves to get the second pair of equations.

15. On the Choice of Axes.

When the problem involves :

Two lines at rt. angles.	Take them as axes.
Two oblique lines.	Use oblique coordinates, or take one line as axis of x, O at intersection.
Two points.	Their join as x-axis, their centre as O.

Triangle.	Oblique axes; one side and its median, *or* One side and its perpendicular.
Square.	Two lines of symmetry (medians or diagonals), *or* Two intersecting sides.
Line and point.	The line and the perpendicular on it from the point.
Point and circle.	Origin at centre of circle, x-axis through the point.
Line and circle.	Origin at centre of circle, x-axis perpendicular to line.
Two circles.	Line of centres and radical axis [see § 22 (6)], *or* Line of centres and perpendicular to it through a centre, *or* A common tangent and $\perp^r$ to it through a centre, *or* Two common tangents.

TRANSFORMATION OF COORDINATES

16. To change to parallel axes through (h, k):

$$x = h + x_1, \quad y = k + y_1.$$

To rotate the axes through $\angle\alpha$ keeping origin unchanged:

$$x = x' \cos\alpha - y' \sin\alpha,$$
$$y = x' \sin\alpha + y' \cos\alpha.$$

Note.—These need scarcely be remembered, for it is easy to reproduce them mentally by the proof: $x = r\cos\theta = r\cos(\theta' + \alpha) =$, &c.

17. Invariants of Transformation. [See also § 54.]

$\dfrac{a + b - 2h\cos\omega}{\sin^2\omega}$, $\dfrac{ab - h^2}{\sin^2\omega}$ are unaltered by any transformation of axes.

Proof.—Consider the condition that

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + 2xy\cos\omega + y^2)$$

should be a square.

THE CIRCLE

18. Forms of Equation.

(1) $x^2+y^2=a^2$. Centre at O, radius a.

(2) $x^2+y^2=2ax$. O on circumference, radius a.

(3) $(x-h)^2+(y-k)^2=a^2$. O at (h, k), radius a.

(4) $x^2+y^2+2gx+2fy+c=0$.

To find centre and radius mentally complete squares.

(5) $x^2+2xy\cos\omega+y^2+2gx+2fy+c=0$.

General equation, axes oblique. Note that (4) cannot represent a circle unless $\omega=90°$.

(6) $x=a\cos\phi$, $y=a\sin\phi$.

The two together constitute the circle, $\phi=\angle POx$ is known as the Auxiliary Angle.

(7) Circle on $(x_1, y_1)(x_2, y_2)$ as diameter.

Express that lines $(x, y)(x_1, y_1)$ and $(x, y)(x_2, y_2)$ are at right angles.

[See also *Polar Equations* (§ 21).]

19. Tangent. [Note.—*It is recommended to learn at this stage the tangent for the General Equation*, § 52.]

(1) $xx_1+yy_1=a^2$.

(2) $x\cos\phi+y\sin\phi=a$. Touches the circle at the point ϕ [§ 18 (6)].

(3) $y=mx\pm a\sqrt{1+m^2}$. Thus the condition that $y=mx+c$ should touch the circle is $c=\pm a\sqrt{1+m^2}$.

(4) The tangent to $(x-h)^2+(y-k)^2=a^2$ is

$$(x-h)(x_1-h)+(y-k)(y_1-k)=a^2,$$

or $(x-h)\cos\phi+(y-k)\sin\phi=a^2$.

(5) The square of the *length of the tangent* from (x_1, y_1) to $f(x, y)=x^2+y^2+2gx+2fy+c=0$ is $T^2=f(x_1, y_1)$.

(6) The equation to the *two tangents* from (x_1, y_1) is

$$(x^2+y^2-a^2)(x_1^2+y_1^2-a^2)=(xx_1+yy_1-a^2)^2.$$

Mnemonic. (Curve)(Curve) = (Tangent)^2.

(7) *Common Tangent* to two circles. See § 23 (2).

20. Pole and Polar.

(1) The polar of (x_1, y_1) with regard to $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2.$$

(2) The polar of (x_1, y_1) with regard to $(x-h)^2+(y-k)^2=a^2$ is $(x-h)(x_1-h)+(y-k)(y_1-k)=a^2$.

Notes.—(1) For all equations of the *second* degree polar and tangent are the same in form.

(2) The equation $xx_1+yy_1=a^2$ has the following interpretations:

α. (x_1, y_1) *outside* the curve: polar, chord of contact of tangents from (x_1, y_1).

β. (x_1, y_1) *on* the curve: tangent.

γ. (x_1, y_1) *inside* the curve: polar.

(3) To determine the pole of $y=mx+c$, identify this and $xx_1+yy_1=a^2$ and thus obtain sufficient equations for x_1 and y_1.

(4) The pole of the chord ϕ_1, ϕ_2 is

$$x_1=a\cos\frac{\phi_1+\phi}{2}\Big/\cos\frac{\phi_1-\phi_2}{2},\quad y_1=a\sin\frac{\phi_1+\phi_2}{2}\Big/\cos\frac{\phi_1-\phi_2}{2}.$$

(5) If the point and its polar with regard to a circle of radius a be distant respectively d and p from the centre,

$$dp=a^2.$$

21. Polar Equations.

(1) Pole at centre: $r=a$.

(2) Pole on circumference:

$r=2a\cos\theta$. [Initial line a diameter.]

$r=2a\cos(\theta-\alpha)$. [Initial line making $\angle\alpha$ with a diameter.]

(3) General equation: $a^2=r^2+l^2-2rl\cos(\theta-\alpha)$ [Centre at (l, α), radius a].

(4) *Tangent*: should be found geometrically as required.

Note.—The tangent at α to $r=2a\cos\theta$ is $2a\cos^2\alpha=r\cos(\theta-2\alpha)$.

22. Radical Axis, Coaxal Circles.

(1) If $S_1=0$, $S_2=0$ be the equations to two circles (in form $x^2+y^2+\ldots=0$), $S_1-S_2=0$ is their radical axis.

(2) The radical axes of the three circles S_1, S_2, S_3 taken in pairs pass through a *radical centre* given by $S_1 = S_2 = S_3$.

(3) $S_1 + \lambda S_2 = 0$ represents all circles coaxal with S_1 and S_2.

(4) All circles which have with S_1 the radical axis $u = 0$ are included in $S_1 + \lambda u = 0$.

(5) To find the orthogonal circle of S_1, S_2, S_3, find the radical centre (h, k), and T the length of the tangent from (h, k) to S_1 [§ 19 (5)], then the orthogonal circle is $(x-h)^2+(y-k)^2 = T^2$.

(6) A system of circles having the same radical axis and line of centres (*coaxal system*) is represented by

$$x^2 + y^2 + 2\lambda x + c = 0. \quad (\lambda \text{ variable.})$$

The axis of this system is $x = 0$.

The *limiting points* are $(\pm \sqrt{c}, 0)$.

The *orthogonal system* is $x^2 + y^2 + 2\mu y + c = 0$.

[See also *GEOM.* § 26.]

Note.—In problems involving two circles it is usually best to refer the circles to their R.A. [See § 15.]

23. CENTRE OF SIMILITUDE.

(1) The C.S. divides the line joining the centres internally and externally in the ratio of the radii. For coordinates see § 2 (*a*).

(2) *Common Tangent.* To find the common tangent to two circles:

(*a*) $(x-h_1)\cos\alpha + (y-k_1)\sin\alpha = a_1$ is a tangent to $(x-h_1)^2+(y-k_1)^2 = a_1^2$. Express that this passes through a C.S.

or, (*b*) Express that the $\perp^r$ on it from $(h_2, k_2) = a_2$, and thus determine α.

(3) *Circle of Similitude* of (h_1, k_1, a_1) and (h_2, k_2, a_2) is

$$\frac{(x-h_1)^2+(y-k_1)^2}{a_1^2} = \frac{(x-h_2)^2+(y-k_2)^2}{a_2^2}.$$

[See also *GEOM.* § 27.]

THE PARABOLA

24. Forms of Equation.

(1) $y^2 = 4ax$.

Axes: Axis, tangent at A; semi-latus rectum $2a$.

(2) $(px+qy)^2 = mx+ny+1$.

General equation: See § 51 (5).

(3) $y^2 = 4a'x$.

Oblique axes: Diameter through P, tangent at P.

$$a' = SP = a/\sin^2\omega.$$

(4) $x = a/m^2$, $y = 2a/m$.

The two together constitute the parabola. Here m is the tangent of the inclination to the axis of x of the tangent at any point.

Other systems: (*a*) $x = a\mu^2$, $y = 2a\mu$: gives the simplest formulae.

(*b*) $x = y_1^2/4a$, $y = y_1$.

(5) *Any* parabola can be represented by $\begin{cases} x+r_1y = a_1t^2+b_1t+c_1. \\ x+r_2y = a_2t^2+b_2t+c_2. \end{cases}$

[See also *Intercept Equation* (§ 29) and *Polar Equation* (§ 30).]

25. Chord.

$$m_1m_2x-\tfrac{1}{2}(m_1+m_2)y+a = 0. \qquad x-\tfrac{1}{2}(\mu_1+\mu_2)y+a\mu_1\mu_2 = 0.$$

Mnemonic: The tangent is $m^2x-my+a = 0$ [$x-\mu y+a\mu^2 = 0$]. If the work of the m's be, so to speak, divided between m_1 and m_2 [or μ_1 and μ_2] we get the chord. Cf. § 52.

Pole. $\dfrac{a}{m_1m_2}$, $a\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}\right)$. $\qquad a\mu_1\mu_2$, $a(\mu_1+\mu_2)$.

Mnemonic: Derived similarly from § 24 (4) and § 24 (4) (*a*).

26. Tangent.

(1) $yy_1 = 2a(x+x_1)$.

Cf. Mnemonic in § 52. The same form for $y^2 = 4a'x$.

(2) $y = mx+\dfrac{a}{m}$.

The m has the same meaning as in § 24 (4). The condition that $y = mx+c$ should touch the curve is $mc = a$.

(3) *Two Tangents*:

$$(y^2-4ax)(y_1^2-4ax_1) = \{yy_1-2a(x+x_1)\}^2.$$

Mnemonic: (Curve) (Curve) = (Tangent)^2.

27. NORMAL.

(1) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(2) $m^3 y + m^2(x - 2a) - a = 0.$ $\qquad y + \mu(x - 2a) - a\mu^3 = 0.$

28. POLAR. The equation $yy_1 = 2a(x + x_1)$ again represents the polar of (x_1, y_1) and has the various meanings given in § 20, *Note* (2). The *pole* of $Ax + By + C = 0$ is found by comparison with $yy_1 = 2a(x + x_1)$.

29. INTERCEPT EQUATION. If the curve be referred to two tangents of lengths h, k making an $\angle \omega$,

(1) *Curve*: $\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1.$

(2) *Tangent* at (x_1, y_1): $\frac{x}{\sqrt{hx_1}} + \frac{y}{\sqrt{ky_1}} = 1.$

This is *not* the polar of (x_1, y_1). [See § 20, *Note* (1).]

(3) *Focus*: $hx = ky = h^2k^2/(h^2 + k^2 + 2hk\cos\omega)$.

(4) *Directrix*: $x(h + k\cos\omega) + y(k + h\cos\omega) = hk\cos\omega$.

Proof.—Prove geometrically by similar triangles for focus

$$hx = ky = SO^2 = x^2 + y^2 + 2xy\cos\omega.$$

The directrix is polar of focus.

30. POLAR EQUATION.

(1) *Curve*: $r = \frac{2a}{1 + \cos\theta} = a\sec^2\frac{\theta}{2}$

if prime vector cuts curve.

$$r = \frac{2a}{1 - \cos\theta} = a\operatorname{cosec}^2\frac{\theta}{2}$$

if prime vector produced backwards cuts curve.

(2) *Chord* $(\alpha + \beta, \alpha - \beta)$: $r\{\cos\theta + \sec\beta\cos(\theta - \alpha)\} = 2a.$

(3) *Tangent at* α: $r\cos\frac{\alpha}{2}\cos\left(\theta - \frac{\alpha}{2}\right) = a.$

(4) *Normal at* α: $r\cos\frac{\alpha}{2}\sin\left(\theta - \frac{\alpha}{2}\right) = a\tan\frac{\alpha}{2}.$

Note.—These equations refer to the first form of equation. To obtain equations for the second form write $\pi - \theta$ for θ, $\pi - \alpha$ for α, &c.

Proofs.—Of (2): Assume chord $r\{A\cos\theta + B\cos(\theta-\alpha)\} = 2a$.
Of (4): See § 11 (3).

(5) *Polar* of (ρ, α): $\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\alpha\right) = \cos(\theta-\alpha)$.

Note.—If any results are remembered here they should be derived from the general conic by putting $e = 1$ in § 39.

31. Magnitude of Related Lines.

(1) $SP = PM = a + x$
$= ST = SG$.

(2) $NG = PK = 2a$,
$PG = 2SY$.

(3) $SY^2 = SA \,.\, SP$
$= a(a+x)$, i.e. $p^2 = ar$.

(4) $\rho = \dfrac{2SP^2}{SY}$.

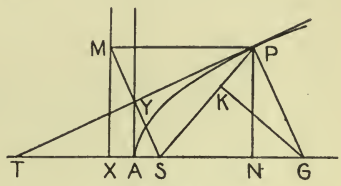


Fig 44

32. Curvature.

(1) *Radius*: $\rho = 2a/\sin^3\phi$, where $\tan\phi = m = 1/\mu$.

Proof.—$\rho = r\dfrac{dr}{dp}$, $p^2 = ar$, $\therefore\ \rho = 2pr/a = 2r^2/p$; but $p = a/\sin\phi$, $r = a/\sin^2\phi$. Or from (2), below.

(2) *Centre of Curvature* at μ: $(2a + 3a\mu^2, -2a\mu^3)$.

Proof.—Find intersection of normals at μ, ν and make $\mu = \nu$.

(3) *Circle of Curvature*: $x^2 + y^2 - 2ax(3\mu^2+2) + 4ay\mu^3 - 3a^2\mu^4 = 0$.
The circle of curvature meets the curve again at the point (-3μ).

(4) *Evolute*: $27ay^2 = 4(x-2a)^3$.

ELLIPSE

33. Forms of Equation.

(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(2) $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} = 1$.

Referred to conjugate diameters $(2a', 2b')$. The angle ω is given by $a'b'\sin\omega = ab$. Also $a'^2 + b'^2 = a^2 + b^2$.

(3) $x^2 + y^2 = c^2$.

Referred to the equiconjugate diameters, $c^2 = \frac{1}{2}(a^2+b^2)$, $\omega = 2\tan^{-1}\frac{b}{a}$. Many of the circle formulae hold for this form.

(4) $x = a\cos\phi$, $y = b\sin\phi$.

The two together constitute the ellipse. ϕ is the 'eccentric' angle $P'CA$, where P' is the point on the auxiliary circle corresponding to P. This auxiliary substitution may also be used for § 33 (2) and the equations for chord, tangent, pole, polar hold, but not that for normal.

[For Polar Equation, see § 39.]

34. Chord.

(1) $$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1+\phi_2}{2} = \cos\frac{\phi_1-\phi_2}{2}.$$

(2) Chord $(x_1, y_1)(x_2, y_2)$ is easily remembered in form

$$\frac{(x-x_1)(x-x_2)}{a^2} + \frac{(y-y_1)(y-y_2)}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

which reduces to

$$\frac{x(x_1+x_2)}{a^2} + \frac{y(y_1+y_2)}{b^2} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + 1.$$

35. Tangent.

(1) $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

(2) $$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

(3) $$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

The condition that $y = mx + c$ should touch the curve is $c^2 = a^2m^2 + b^2$.

(4) $$x\cos\alpha + y\sin\alpha = \pm\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

So that if p be the central perpendicular on the tangent $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.

(5) *Two tangents* from (x_1, y_1):

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

Mnemonic: (Curve)(Curve) = (Tangent)^2.

(6) *Director circle*, i.e. locus of intersection of perpendicular tangents: $x^2 + y^2 = a^2 + b^2$.

36. Normal.

(1) $$\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}.$$

Not so much learnt as written immediately from § 35 (1).

(2) $$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

(3) $(y - mx)^2 (a^2 + m^2 b^2) = (a^2 - b^2)^2 m^2$.

37. Polar of (x_1, y_1). The equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ again represents the polar of (x_1, y_1) and has the various meanings given in § 20, *Note* (2). The pole of $Ax + By + C = 0$ is found by comparison with it.

Pole of chord ϕ_1, ϕ_2: $$\frac{a\cos\frac{\phi_1 + \phi_2}{2}}{\cos\frac{\phi_1 - \phi_2}{2}}, \quad \frac{b\sin\frac{\phi_1 + \phi_2}{2}}{\cos\frac{\phi_1 - \phi_2}{2}}.$$

38. Conjugate Diameters ($2a', 2b'$).

(1) $a'^2 + b'^2 = a^2 + b^2$; $a'b' \sin\omega = ab$.

(2) If α, α' be the inclinations to the x-axis of two conjugate diameters,

$$\tan\alpha \,.\, \tan\alpha' = -\frac{b^2}{a^2}.$$

(3) If ϕ, ϕ' be the eccentric angles of the extremities of two conjugate diameters,

$$\phi' - \phi = (\text{an odd multiple of}) \frac{\pi}{2}.$$

(4) The *equiconjugate diameters* are inclined to the x-axis at an angle $\tan^{-1}\frac{b}{a}$. [From § 38 (2), put $\alpha' = \pi - \alpha$.]

(5) If (x_1, y_1) (x_2, y_2) be the extremities of any two conjugate diameters, the properties overleaf belong to the determinant: $\begin{vmatrix} \frac{x_1}{a} & \frac{x_2}{a} \\ \frac{y_1}{b} & \frac{y_2}{b} \end{vmatrix}$

(α) Sum of squares of any row or column $= 1$.

(β) Sum of products of rows or columns $= 0$.

(γ) Whole determinant $= \pm 1$.

39. Polar Equations.

(1) *Curve*: $r = \dfrac{l}{1+e\cos\theta}$ $\left[or\ \dfrac{l}{1-e\cos\theta}\right]$.

(2) *Chord* $(\alpha+\beta, \alpha-\beta)$: $r\{e\cos\theta+\sec\beta\,.\cos(\theta-\alpha)\} = l$.

(3) *Tangent at* α: $r\{e\cos\theta+\cos(\theta-\alpha)\} = l$.

(4) *Normal at* α: $\dfrac{l}{r}\cdot\dfrac{e\sin\alpha}{1+e\cos\alpha} = \sin(\theta-\alpha)+e\sin\theta$.

Proofs.—Of (2): Assume $r[A\cos\theta+B\cos(\theta-\alpha)] = l$. Of (4): See § 11 (3), or it can be easily derived directly from $SG = e\,.\,SP$.

(5) *Polar of* (ρ, α): $\left(\dfrac{l}{r}-e\cos\theta\right)\left(\dfrac{l}{\rho}-e\cos\alpha\right) = \cos(\theta-\alpha)$.

40. Concyclic and Conormal Points. If the eccentric angles of four points be α, β, γ, δ.

(1) If α, β, γ, δ are concyclic, $\alpha+\beta+\gamma+\delta = 2n\pi$, *and conversely.*

Hence (i) the lines (α, β) (γ, δ) are equally inclined to either axis.

(ii) The point common to an ellipse and its circle of curvature. Put $\alpha = \beta = \gamma$, &c.

(2) If α, β, γ, δ are conormal, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$, but *the converse is not true.*

(3) If α, β, γ are conormal, $\Sigma\sin(\beta+\gamma) = 0$, *and conversely.*

41. Magnitude of Related Lines.

(1) $BC = b$, $AC = a$, $e^2 = 1 - \dfrac{b^2}{a^2}$.

(2) $CS = ae$, $CA = a$, $CX = \dfrac{a}{e}$.

(3) Semi-latus rectum $(l) = \dfrac{b^2}{a} = a(1-e^2)$.

(4) $SP = a + ex$, $S'P = a - ex$, $SP + S'P = 2a$,

$$SP \,.\, S'P = CD^2.$$

(5) $PG = \frac{b}{a} CD$, $CG = e^2 x$ [hence NG].

(6) $SY \,.\, S'Z = BC^2$, PG bisects $\angle SPS'$.

(7) If $SY = p$, $SP = r$, then $\frac{b^2}{p^2} = \frac{2a}{r} - 1$.

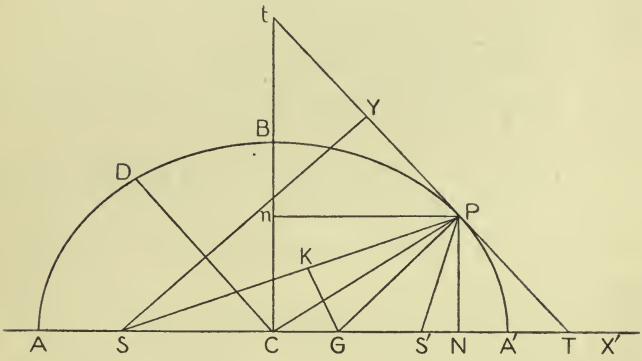


Fig. 45.

(8) If p be the perpendicular from C on the tangent at P, $p \,.\, CD = ab$, $CP^2 + CD^2 = a^2 + b^2$, $CP \,.\, CD \sin \angle PCD = ab$.

(9) $SG = ae + e^2 x = e \,.\, SP$. Hence $\triangle SKG$ ||| to SNP, the ratio of linear dimensions being $e : 1$, $\therefore GK = e \,.\, PN = ey$. Hence $\angle SPG$ is known. $PK = l$.

(10) $CT = \frac{a^2}{x}$, $Ct = \frac{b^2}{y}$.

(11) $\rho = CD^2/p = CD^3/ab = a^2b^2/p^3$, where p is the central $\perp^{\text{r}}$ on the tangent. It can be found in terms of SY, SP from (7) by $\rho = r\frac{dr}{dp}$.

42. Curvature.

(1) *Radius*: $\rho = \dfrac{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{ab} = \dfrac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}$. [Origin *C*.]

[See also § 41 (11).]

(2) *Centre of Curvature*: $\left(\dfrac{a^2-b^2}{a}\cos^3\phi,\ \dfrac{b^2-a^2}{b}\sin^3\phi\right)$.

(3) *Evolute*: $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

THE HYPERBOLA

43. Analogy with Ellipse.

Rule i. Any ellipse formula involving even powers of b or e only becomes the corresponding hyperbola formula on writing $-b^2$ for $+b^2$.

Rule ii. Ellipse formulae involving odd powers of b or e as a rule require total modification.

Rule iii. Polar equations apply unchanged.

Exx.: *Equation*: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Eccentricity: $e^2 = 1 + \dfrac{b^2}{a^2}$.

Tangent: $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$; $y = mx + \sqrt{a^2m^2 - b^2}$.

Conjugate Diameters: $\tan\alpha \,.\, \tan\alpha' = b^2/a^2$.

Director Circle: $x^2 + y^2 = a^2 - b^2$, &c.

44. The Auxiliary Angle is not of much service in the case of the hyperbola referred to its axes. Various systems exist and a few formulae are given for reference, but none should be learnt.

(1) $x = a\sec\phi$, $y = b\tan\phi$. Here ϕ has a geometrical interpretation. Let the tan. at P meet the major axis in T, and TQ drawn $\perp^r$ to this axis meet the auxiliary circle in Q, then $\angle QCT = \phi$.

(2) $x = a\cosh\phi$, $y = b\sinh\phi$.

(3) $x = a\cos\phi$, $y = ib\sin\phi$. This system is on the whole recommended, for it admits of any formula being written from the corresponding ellipse formula. Great care is needed in the interpretation of results. ϕ is an imaginary angle. The relation $\phi' - \phi = \dfrac{\pi}{2}$ is still true for the extremities of conjugate diameters, though ϕ and ϕ' are

both imaginary. The point thus obtained as conjugate to P is not however D, but an imaginary point distant from the centre i. CD and really on the hyperbola, not on its conjugate.

(4) *Formulae*:

	(*a*) $x = a\sec\phi,\ y = b\tan\phi.$	(*b*) $x = a\cosh\phi,\ y = b\sinh\phi.$
Chord ϕ_1, ϕ_2.	$\frac{x}{a}\cos\frac{\phi_1-\phi_2}{2} - \frac{y}{b}\sin\frac{\phi_1+\phi_2}{2} = \cos\frac{\phi_1+\phi_2}{2}.$	$\frac{x}{a}\cosh\frac{\phi_1+\phi_2}{2} - \frac{y}{b}\sinh\frac{\phi_1+\phi_2}{2} = \cosh\frac{\phi_1-\phi_2}{2}.$
Tangent at ϕ.	$\frac{x}{a} - \frac{y}{b}\sin\phi = \cos\phi.$	$\frac{x}{a}\cosh\phi - \frac{y}{b}\sinh\phi = 1.$
Normal ϕ.	$ax\cos\phi + by\cot\phi = a^2+b^2.$	$\frac{ax}{\cosh\phi} + \frac{by}{\sinh\phi} = a^2+b^2.$
Pole ϕ_1, ϕ_2.	$\frac{a\cos\frac{\phi_1-\phi_2}{2}}{\cos\frac{\phi_1+\phi_2}{2}},\ b\tan\frac{\phi_1+\phi_2}{2}$	$\frac{a\cosh\frac{\phi_1+\phi_2}{2}}{\cosh\frac{\phi_1-\phi_2}{2}},\ \frac{b\sinh\frac{\phi_1+\phi_2}{2}}{\cosh\frac{\phi_1-\phi_2}{2}}.$

45. The Conjugate Hyperbola. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$

Note.—The point conjugate to P on the original hyperbola, i.e. such that the tangent at it is $\parallel CP$ and the vector to it $\parallel$ to tangent at P, is imaginary. A real point D is taken on a line through $C \parallel$ tangent at P and such that i. CD = the distance from C of the true conjugate of P. The locus of D is the conjugate hyperbola and D is called the point conjugate to P. The following properties are true concerning it:

(1) $CP^2 - CD^2 = a^2 - b^2$.

(2) Tangents at P and D intersect on the asymptotes.

(3) Area CPD is constant. [See also *G. CON.* § 25.]

Note.—Thus the minor axis of the hyperbola is not really CB but i. CB.

46. Magnitude of Related Lines.

(1) $e^2 = 1 + \frac{b^2}{a^2}$.

(2) $CS = ae,\ CA = a,\ CX = a/e$.

(3) Semi-latus rectum $(l) = b^2/a = a(e^2 - 1)$.

(4) $SP = ex - a$, $S'P = ex + a$, $S'P - SP = 2a$,

$$SP \,.\, S'P = CD^2.$$

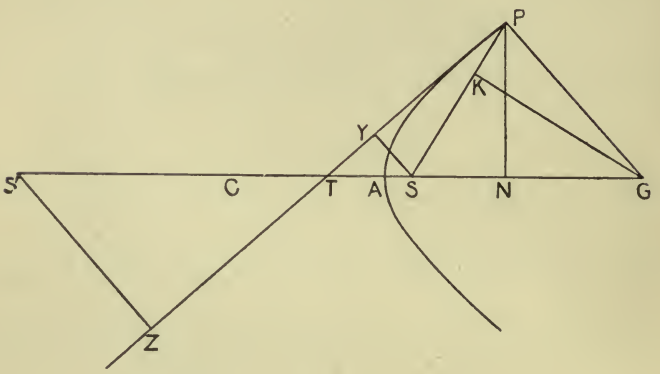


Fig. 46.

(5) $PG = \frac{b}{a} \,.\, CD$, $CG = e^2 x$. [Hence NG.]

(6) $SY \,.\, S'Z = BC^2$ and PT bisects $\angle SPS'$.

(7) If $SY = p$, $SP = r$, $\frac{b^2}{p^2} = \frac{2a}{r} + 1$.

Note.— $\frac{b^2}{p^2} = \frac{2a}{r} \begin{matrix} -1 \\ +0 \\ +1 \end{matrix}$ as the curve is an ellipse, parabola, or hyperbola.

(8) If p be the perpendicular from C on the tangent at P, $p \,.\, CD = ab$, $CP^2 - CD^2 = a^2 - b^2$, $CP \,.\, CD \sin \angle PCD = ab$, and the area of the Δ cut off from the asymptotes by any tangent is constant.

(9) $SG = e^2 x - ae = e \,.\, SP$. Hence $\Delta SKG \,|||\, \Delta SNP$, the ratio of linear dimensions being $e : 1$, $\therefore$ $GK = e \,.\, PN = ey$. Hence $\angle SPG$. $PK = l$.

(10) $CT = a^2/x$, $Ct = b^2/y$.

(11) Angle between the asymptotes $= 2 \tan^{-1} b/a$.

(12) $\rho = CD^2/p = CD^3/ab = a^2b^2/p^3$, where p is the central $\perp^r$ on the tangent.

47. Hyperbola referred to Asymptotes (axes oblique).

(1) *Curve*: $xy = \frac{1}{4}(a^2+b^2) = c^2$.

(2) *Tangent*: $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$.

Note.—The polar of (x_1, y_1) is *not* of this form. It is $xy_1 + yx_1 = 2c^2$ and *not* $xy_1 + yx_1 = 2x_1y_1$.

(3) *Chord* $(x_1, y_1)(x_2, y_2)$: $\dfrac{x}{x_1+x_2} + \dfrac{y}{y_1+y_2} = 1$.

[Cf. *Mnemonic* in § 25.]

Note.—It is very useful to note that in this form any point may be denoted by $(ct, c/t)$. The following equations then hold:

Tangent at t: $x + yt^2 = 2ct$.

Chord (t_1, t_2): $x + yt_1t_2 = c(t_1+t_2)$. [*Mnemonic*, § 25.]

Pole of Chord: $2ct_1t_2/(t_1+t_2),\ 2c/(t_1+t_2)$.

48. Rectangular Hyperbola (= *Equilateral Hyperbola*).

(1) $a = b$, $e^2 = 1 + \dfrac{b^2}{a^2} = 2$. Conjugate also rectangular.

(2) *Curve*: $x^2 - y^2 = a^2$ (referred to axes),

$$xy = c^2 = \frac{a^2}{2} \text{ (referred to asymptotes).}$$

Note.—Of these equations to the curve the second is most useful and the *note* of § 47 becomes exceedingly valuable. The following formulae may be added:

Normal at t: $yt - xt^3 + c(t^4-1) = 0$.

Centre of Curvature at t: $\{c(3t^4+1)/2t^3,\ c(t^4+3)/2t\}$.

Radius of Curvature at t: $c(t^4+1)^{\frac{3}{2}}/2t^3$.

The normal meets the curve again in the point $-1/t^3$.

The circle of curvature meets the curve again in the point $1/t^3$.

49. CURVATURE. [See also § 46 (12) and § 48, *Note.*]

(1) *Radius*: $\rho = \dfrac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}$.

(2) *Centre of Curvature*: $ax = (a^2+b^2)\sec^3\phi$, $by = -(a^2+b^2)\tan^3\phi$. [§ 44 (*a*).]

$ax = (a^2+b^2)\cosh^3\phi$, $by = -(a^2+b^2)\sinh^3\phi$. [§ 44 (*b*).]

(3) *Evolute*: $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$.

SOME GENERAL PRINCIPLES

50. If $\alpha = 0$, $\beta = 0$ ... represent straight lines and $S_1 = 0$, $S_2 = 0$ ... conics,

(1) $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ pass through a point if it is possible to determine m and n so that $S_1 + mS_2 + nS_3 = 0$ identically.

e. g. $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$ pass through a point because $(S_1 - S_2) + (S_2 - S_3) + (S_3 - S_1) = 0$ identically.

(2) Any conic through all the intersections of $S_1 = 0$, $S_2 = 0$ may be written $S_1 + kS_2 = 0$.

(3) Any conic through the points where $\alpha = 0$ meets $S_1 = 0$ may be written $S_1 + (lx + my + n)\alpha = 0$. ($l$, m, n arbitrary constants.)

(4) Any conic circumscribing the triangle $\alpha = 0$, $\beta = 0$, $\gamma = 0$ can be written as $p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$.

(5) Any conic circumscribing the quad[1] $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ can be written as $\alpha\gamma = k\,.\,\beta\delta$.

Notes. — (1) If $\alpha \equiv x\cos\alpha + y\sin\alpha - p = 0$ the circumcircle is $\alpha\gamma = \pm k\beta\delta$, the upper sign being taken if O is *outside* the quadrilateral.

(2) Any circle may be written

$$(x+iy)(x-iy) + (2gx + 2fy + c)(0\,.\,x + 0\,.\,y + 1) = 0$$

and therefore passes through the points at infinity (*Focoids*) given by $x \pm iy = 0$, $0\,.\,x + 0\,.\,y + 1 = 0$.

(6) To find the ratio in which the line (x, y) (x_1, y_1) is cut by the curve $f(x, y) = 0$ solve in n/n_1 the equation

$$f\left[\frac{nx_1 + n_1x}{n + n_1}, \frac{ny_1 + n_1y}{n + n_1}\right] = 0.$$

(7) To find the equation to all the tangents from (x_1, y_1) to $f(x, y) = 0$ express that the equation in (6) has equal roots.

(8) To find the equation of lines drawn from (x_1, y_1) to all points of intersection of two curves substitute $\frac{kx_1 + x}{k+1}$, $\frac{ky_1 + y}{k+1}$ for x and y in both curves and eliminate k.

(9) To find the length of the line drawn from (x_1, y_1) in a given direction θ to meet the curve $f(x, y) = 0$ solve $f(x_1 + r\cos\theta, y_1 + r\sin\theta) = 0$.

Applications:

(1) To find locus of midpoints of ‖ chords. Make roots equal and opposite.

(2) To find directions or equations of two tangents. Make roots equal.

(10) *Five* points completely determine an ellipse or hyperbola.

Four points completely determine a parabola.
Three ,, ,, ,, a circle.
Two ,, ,, ,, a st. line.

Instead of one or more of these points may be given an equal number of tangents to the curve, but in this case there will generally be several curves satisfying the conditions.

GENERAL EQUATION OF SECOND DEGREE

51. Analysis. The equation is taken as

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (a, b, c, f, g, h)(x, y, 1)^2 = 0.$$

(1) If $h = 0$ and $a = b$ it represents a circle. [See § 18 (4).]

If the axes are oblique $h = a\cos\omega$ and $a = b$.

(2) Then determine $h^2 - ab$.

If this is $-$ the curve is an ellipse [circle, or point].

If this is 0 the curve is a parabola [two coincident or parallel lines].

If this is $+$ the curve is a hyperbola [two st. lines].

(3) If $h^2 - ab$ is + or 0, test for two st. lines by factorization. [§ 12 (3).]

(4) If it proves not to represent two st. lines and $h^2 - ab$ is not zero, the curve is a central conic (ellipse or hyperbola).

(*a*) *Find the Centre.* Write $x + X$ for x, $y + Y$ for y, and equate coefficients of x, y to zero. Solve for (X, Y) the centre.

Or, Solve $\frac{\delta S}{\delta x} = 0, \frac{\delta S}{\delta y} = 0.$

Note.—If either of the coordinates x, y is ∞, the curve is a parabola; if indeterminate, two parallel lines.

(*b*) *Refer to Centre* (§ 16). The result referred to ∥ axes through the centre is found by rule: *Leave unaltered the terms of the second degree and the absolute term and write HALF the coordinates of the centre for x and y in the terms of the first degree.*

The curve is now $ax^2 + 2hxy + by^2 = c'$.

Note.—If two st. lines have been missed at § 51 (3) they will now reveal themselves by $c' = 0$.

(*c*) *Position of Principal Axes.* They are inclined to axis of x at angles α (acute) and $\frac{\pi}{2} + \alpha$, which satisfy the equation

$$\tan 2\theta = \frac{2h}{a - b}.$$

Note.—Can be derived by § 12 (2), the asymptotes being

$$ax^2 + 2hxy + by^2 = 0.$$

(*d*) *Lengths of Principal Axes.* If the equation referred to them is $Ax^2 + By^2 = c'$, determine A and B by

$$AB = ab - h^2,$$
$$A + B = a + b.$$

[For Oblique Coordinates, § 17.]

(*e*) *Discrimination of Principal Axes.* To determine whether the major or the minor axis lies along the line $\theta = \alpha$ (acute) if the curve is

A Hyperbola: Draw a rough figure, remembering that the asymptotes are $ax^2 + 2hxy + by^2 = 0$.

An Ellipse: It will be the *major* axis *if c' and h have opposite signs.*

Or, The lines joining O to the intersection of $x^2+y^2 = r^2$ and $ax^2+2hxy+by^2 = c'$ are $\left(a - \frac{c'}{r^2}\right)x^2+2hxy+\left(b-\frac{c'}{r^2}\right)y^2 = 0$. These coincide if r is a semi-axis. Hence semi-axes satisfy

$$\left(a - \frac{c'}{r^2}\right)\left(b - \frac{c'}{r^2}\right) = h^2,$$

and the equation to the semi-axis of length r is $\left(a - \frac{c'}{r^2}\right)x + hy = 0$. This method is applicable to oblique coordinates, writing

$$x^2+2xy\cos\omega+y^2 = r^2, \text{ \&c.}$$

(5) If it proves not to represent two st. lines and $h^2 - ab$ is zero it is a parabola. In this case it may be written

$$(px + qy)^2 = mx + ny + 1;$$

i.e. $(px + qy + k)^2 = (m + 2pk)x + (n + 2qk)y + 1 + k^2$.

Choose k so that the lines

$$px + qy + k = 0,$$

$$(m + 2pk)x + (n + 2qk)y + 1 + k^2 = 0$$

are at rt. angles and take them for axes of x and y respectively. By finding Y, X, the perpendiculars on these lines, the equation is written in the form $Y^2 = \pm AX$.

Note.—The ambiguity is best determined by a rough figure, but see § 9, *Note.*

52. TANGENT at x_1, y_1 to $S = 0$ is

$$T = axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Mnemonic.—Divide the work of the x's in S, so to speak, between x and x_1, and the work of the y's between y and y_1. Cf. § 25.

Polar of (x_1, y_1): Same form as $T = 0$.

Chord whose middle point is (x_1, y_1): $T = S_1$, where S_1 is the result of substituting x_1, y_1 for x, y in S.

Two tangents from (x_1, y_1): $SS_1 = T^2$.

Mnemonic: (Curve)(Curve) = (Tangent)^2.

53. General Results. $S = (a, b, c, f, g, h)(x, y, 1)^2 = 0.$

If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and $A, B, C, \ldots$ are the minors of $a, b, c, \ldots,$

(1) *Tangent* at $(X, Y, Z = 1)$
Polar " " " $\Big\}\ x\frac{\delta S}{\delta X} + y\frac{\delta S}{\delta Y} + z\frac{\delta S}{\delta Z} = 0.$

Two Tangents from (X, Y, Z):

$$4S(x, y, z)\,S(X, Y, Z) = \left(x\frac{\delta S}{\delta X} + y\frac{\delta S}{\delta Y} + z\frac{\delta S}{\delta Z}\right)^2.$$

(2) *Centre*: $S_x = 0,\ S_y = 0.$

These lead to $x/G = y/F = 1/C.$

(3) *Foci*: $\dfrac{S_x^2 - S_y^2}{a - b} = \dfrac{2S_x S_y}{2h} = 4S.$ [Cf. 12 (2).]

These work out to $\begin{cases} C(x^2 - y^2) - 2Gx + 2Fy + A - B = 0, \\ Cxy - Fx - Gy + H = 0. \end{cases}$

These two R.H.'s which intersect in the foci are the loci of points the two tangents from which to the conic make complementary and supplementary angles with the axis of x respectively.

Proof.—Express that two tangents [§ 52] from foci satisfy the conditions for a circle [§ 55, *Note* 2].

Note.—The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has four foci $(\pm ae, 0)\ (0, \pm aei)$ and two eccentricities

$$\sqrt{1 - \frac{b^2}{a^2}},\quad \sqrt{1 - \frac{a^2}{b^2}}.$$

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has four foci $(\pm ae, 0)\ (0, \pm aei)$ and two eccentricities, both real,

$$\sqrt{1 + \frac{b^2}{a^2}},\quad \sqrt{1 + \frac{a^2}{b^2}}.$$

(4) *Axes*: $\dfrac{S_x^2 - S_y^2}{a - b} = \dfrac{2S_x S_y}{2h}.$

(5) *Asymptotes*: $aS_y^2 + bS_x^2 - 2hS_y S_x = 0,$
or $CS = \Delta.$

Conjugate Hyperbola: $CS = 2\Delta.$

(6) *Director Circle*: $C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0.$

For the directrix of the general parabola put $C = 0.$

(7) *Eccentricity*: $e^4 + \dfrac{(a-b)^2+4h^2}{ab-h^2}(e^2-1) = 0.$

(8) *Axes* $(2r)$: $C^2r^4 + C\Delta(a+b)r^2 + \Delta^2 = 0.$

(9) $lx+my+n = 0$ touches S if

$$(A, B, C, F, G, H)(l, m, n)^2 = 0.$$

Note.—This may also be written by equating to zero the determinant formed by 'bordering' Δ with l, m, n, and zero.

(10) $y = mx$, $y = m'x$ are parallel to conjugate diameters if

$$a + h(m+m') + bmm' = 0.$$

54. Invariants.

I. *Of a single Conic*:

(1) When the axes are rectangular, to turn them through any angle whatever does not affect the values of

$$ab-h^2,\ a+b,\ f^2+g^2,\ c,\ \Delta.$$

(2) When the axes are oblique no transformation affects the values of $\dfrac{ab-h^2}{\sin^2\omega}$ and $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$. [Proof: § 17].

II. *Of two Conics*:

If $S = 0$, $S' = 0$ be two conics, $S+\lambda S' = 0$ will be a pair of straight lines if

$$\begin{vmatrix} a+\lambda a', & h+\lambda h', & g+\lambda g' \\ h+\lambda h', & b+\lambda b', & f+\lambda f' \\ g+\lambda g', & f+\lambda f', & c+\lambda c' \end{vmatrix} = 0.$$

If this equation be written $\Delta+\lambda\Theta+\lambda^2\Theta'+\lambda^3\Delta' = 0$, the ratios $\Delta:\Theta:\Theta':\Delta'$ are unaffected by any change of axes.

$$\Theta = Aa'+Bb'+Cc'+2Ff'+2Gg'+2Hh';$$
$$\Theta' = A'a+B'b+C'c+2F'f+2G'g+2H'h.$$

(1) The condition (necessary and sufficient) that the conics should touch is that the λ-equation should have two equal roots.

(2) The condition (necessary and sufficient) that the conics should osculate is that the λ-equation should have three equal roots.

(3) The condition (necessary and sufficient) that the conics should be such that a Δ circumscribed to S will be inscribed in S' is $\Theta^2 = 4\Delta\Theta'$. If there is one such Δ there is any infinite number of such.

PARTICULAR CONICS

55. Conic $\beta\gamma = k\alpha^2$ touches $\beta = 0$, $\gamma = 0$, where $\alpha = 0$ meets them.

Notes.—(1) Any parabola may be written $y^2 = (0 \cdot x + 0 \cdot y + 4a)x$, $\therefore$ touches line at ∞.

(2) If the origin is at the focus and $D = 0$ be a directrix, the conic is $x^2+y^2 = \lambda D^2$, $\therefore$ $x+yi = 0, x-yi = 0$ are two tangents from focus. These pass through focoids, $\therefore$ all confocal conics have four imaginary common tangents, and two opposite vertices of 4^1 thus formed are foci.

Application.—If a 4^1 whose sides are $L = 0$, $L' = 0$, $M = 0$, $M' = 0$, is circumscribed to conic, and if $R = 0$ joins the contacts of L and M, and $R' = 0$ those of $L'M'$, then R and R' pass through intersection of diagonals of 4^1. Conic may be written $LM - R^2 = 0$, also $L'M' - R'^2 = 0$,

$$\therefore\ LM - R^2 = \lambda(L'M' - R'^2),\ \therefore\ LM - \lambda \cdot L'M' = R^2 - \lambda R'^2,$$

$$\therefore\ LM - \lambda \cdot L'M' = 0$$

is two st. lines, passing through points (L, L') (L, M') (M, L') (M, M'), $\therefore$ the two diagonals. It also passes through (R, R').

56. Conics $S = k\alpha\beta$, $S = k\alpha^2$.

(1) $S = k\alpha\beta$ passes through the four points where $\alpha = 0$, $\beta = 0$ meet $S = 0$.

(2) $S = k\alpha^2$ touches $S = 0$ where $\alpha = 0$ meets it.

Notes.—(1) From § 55, *Note* (2), the focus may be regarded as an infinitely small circle having double contact with the conic, the chord of contact being directrix.

(2) All similar and similarly situated conics have double contact on the line at ∞.

Application.—$S + L^2 = 0$, $S + M^2 = 0$, $S + N^2 = 0$ represent three conics having double contact with $S = 0$. Their chords of intersection are $L^2 - M^2 = 0$, $M^2 - N^2 = 0$, $N^2 - L^2 = 0$, and are concurrent in threes. If the conics $S + L^2 = 0$, &c., are all line-pairs we have Brianchon's Theorem. [See § 65 (3).]

57. Conic through Four Points.

(1) Conic through $(s, 0)$ $(s', 0)$ $(0, t)$ $(0, t')$ is

$$\frac{x^2}{ss'} + 2hxy + \frac{y^2}{tt'} - x\left(\frac{1}{s} + \frac{1}{s'}\right) - y\left(\frac{1}{t} + \frac{1}{t'}\right) + 1 = 0.$$

(2) *Or*, the points may be defined as the points where $ax+by=1$, $a'x+b'y=1$ meet the axes, and the conic is then

$$(ax+by-1)(a'x+b'y-1) = \lambda xy.$$

(3) *Or*, taking any axes, the lines joining the points may be found and § 50 (5) used.

58. Conic touching Axes at $(s, 0)(0, t)$ is

$$\frac{x}{s}+\frac{y}{t}-1 = \sqrt{\lambda xy}.$$

For conic touching two given lines at given points use § 55.

59. Conic touching Four Lines. Take lines as $x=0$, $y=0$, $lx+my=1$, $l'x+m'y=1$. Any conic touching $x=0$, $y=0$ is $(ax+by-1)^2 = 2\lambda xy$. Determine a and b by remaining conditions of contact.

Note.—$\lambda = 2(a-l)(b-m) = 2(a-l')(b-m')$.

60. Concentric Conics may often be best treated by referring them to their common pair of conjugate diameters and writing them

$$ax^2+by^2=1,\ a'x^2+b'y^2=1.$$

61. Confocal Conics.

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$ are confocal if

$$a^2-a'^2 = b^2-b'^2.$$

(2) The family confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

(3) Confocal conics intersect at right angles.

(4) Tangents from P to a conic are equally inclined to the confocal through P.

62. Similar Conics.

(1) Two conics are similar and similarly situated if

$$\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}.$$

(2) Two conics are similar if $\dfrac{(a+b)^2}{h^2-ab} = \dfrac{(a'+b')^2}{h'^2-a'b'}$.

In oblique coordinates $\dfrac{(a+b-2h\cos\omega)^2}{h^2-ab} = \dfrac{(a'+b'-2h'\cos\omega')^2}{h'^2-a'b'}$.

ENVELOPES

63. (1) The envelope of $\mu^2 L + \mu M + N = 0$ is $M^2 = 4LN$.

(2) The envelope of

$$P\cos\alpha + Q\sin\alpha = R \text{ is } P^2 + Q^2 = R^2.$$

[See *DIFF. CALC.* § 24.]

CONTACT OF CONICS

64. (1) Take the conics in the form

$$a x^2 + 2h xy + b y^2 + 2g x = 0, \quad \text{[Contact}$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0. \quad \text{First Order.]}$$

Note.—This form of equation, referring the conic to a tangent ($x=0$) and normal ($y = 0$), is often very convenient and should be carefully noted.

The line through the other two intersections P, Q is

$$(ab' - a'b)x + 2(hb' - h'b)y - 2(bg' - b'g) = 0.$$

If O lies on this line, $bg' - b'g = 0$.

[**Contact Second Order.**]

The equations to OP, OQ are

$$(ag' - a'g)x^2 + 2(hg' - h'g)xy + (bg' - b'g)y^2 = 0.$$

If OP, OQ coincide, $(hg' - h'g)^2 = (ag' - a'g)(bg' - b'g)$.

[**Double Contact.**]

If OP, OQ coincide and O lies on PQ,

$$\frac{h}{h'} = \frac{b}{b'} = \frac{g}{g'}. \quad \text{[Contact Third Order.]}$$

Hence equations to circle of curvature, centre of curvature, ρ, osculating conic, &c. [See §§ 32, 41 (11), 42, 46 (12), 48 (*Note*), 49.]

(2) *Or* thus:

Let $S = 0$ be a conic, (x', y') a point on it, and $T = 0$ the tangent at (x', y').

Then $S - \lambda T\{(y-y') - m(x-x')\} = 0$ is a conic intersecting $S = 0$ in (x', y') at three consecutive points, and m, λ may be chosen so that this is a circle or otherwise.

$S - \lambda T^2 = 0$ has **Contact Third Order.**

SOME GENERAL THEOREMS

65. (1) CARNOT'S THEOREM. If a conic meet the sides of a Δ in a, a'; b, b'; c, c',

$$\frac{Ba \,.\, Ba'}{Ca \,.\, Ca'} \cdot \frac{Cb \,.\, Cb'}{Ab \,.\, Ab'} \cdot \frac{Ac \,.\, Ac'}{Bc \,.\, Bc'} = 1.$$

Mnemonic.—$B/C \,.\, C/A \,.\, A/B = 1$.

Note.—Hence describe a conic, given four points and one tangent.

(2) PASCAL'S THEOREM. If a hexagon be inscribed in a conic the crosses of opposite sides are collinear.

Notes.—(1) There are sixty Pascal Lines corresponding to six points on a conic.

(2) By this theorem, (α) Draw a tangent at a point on conic by ruler only; (β) Describe a conic through five points.

Proof.—If the vertices be a, b, c, d, e, f and $ab = 0$ denote the line ab, then conic may be written $ab \,.\, cd - bc \,.\, ad = 0$ (for it circumscribes a, b, c, d) and also $de \,.\, fa - ef \,.\, ad = 0$ (for it circumscribes d, e, f, a);

$$\therefore\ ab \,.\, cd - de \,.\, fa = ad\,(bc - ef).$$

L. H. of this equation represents conic circumscribing ab, de, cd, fa. But it splits into factors, $\therefore$ is the two diagonals of this 4^1. But ad is one diagonal, $\therefore$ $bc - ef = 0$ joins the vertices (ab, de) (cd, af). But it clearly passes through (bc, ef), $\therefore$ &c.

(3) BRIANCHON'S THEOREM. If a hexagon be described about a conic the joins of opposite angles are concurrent. [See § 56 (2) *Application.*]

Notes.—(1) There are sixty Brianchon Points corresponding to six tangents to a conic.

(2) By this theorem describe a conic touching five lines.

ANHARMONIC RATIO

[See also *GEOM.* §§ 20, 22, 23, 24, 31, and *G. CON.* § 32. Some theorems are repeated here for convenience.]

66. (1) $\{PQRS\} = \dfrac{PQ}{QR} \cdot \dfrac{RS}{SP} = -\dfrac{PQ}{QR}$ if S be at ∞.

$$\{O.PQRS\} = \frac{\sin \angle POQ}{\sin \angle QOR} \cdot \frac{\sin \angle ROS}{\sin \angle SOP} = \{PQRS\}$$

if these points are on a line.

Mnemonic.—Note order *PQ, QR, RS, SP.*

(2) If $\{PQRS\} = -1$, PQ, PR, and PS are in H.P. and we have a Harmonic Range.

(3) If the abscissae of P and R be given by

$$ax^2 + 2hx + b = 0,$$

If the abscissae of Q and S be given by

$$a'x^2 + 2h'x + b' = 0,$$

$\{PQRS\} = -1$ if $ab' + a'b - 2hh' = 0$.

(4) The A.R. of the pencil $y = kx$, $y = lx$, $y = mx$, $y = nx$ is $\dfrac{(k-l)}{(l-m)} \cdot \dfrac{(m-n)}{(n-k)}$, whatever the $\angle$ between the axes, and the same result expresses the A.R. of $\alpha = k\beta$, $\alpha = l\beta$, $\alpha = m\beta$, $\alpha = n\beta$, where $\alpha = 0$, $\beta = 0$ are any two lines.

(5) Each of the three diagonals of a 4^1 is divided harmonically by the other two diagonals.

(6) The A.R. of the pencil formed by joining four fixed points on a conic to any fifth variable point is constant.

(7) The A.R. of the range formed by cutting four fixed tangents to a conic by any fifth variable tangent is constant.

(8) If a line through a point O meet a conic in R and S and meet the polar of O in P, $\{ORPS\} = -1$.

(9) Two tangents through any point, any other line through the point, and the line to the pole of this last line form a Harmonic Pencil.

(10) In deriving theorems from these properties, especially (6) and (7): 'The four points on the curve may be any whatever, and either one or two of them may be at ∞: the fifth point to which the pencil is drawn may be either at an infinite distance, or may coincide with one of the four points, in which latter case one of the legs of the pencil will be the tangent at that point. Then again we may measure the A.R. by the segments on any line drawn across it, which we may, if we please, draw $\parallel$ to one of the legs of the pencil and so reduce the ratio to a simple one.'

67. Homographic Division. [See *GEOM.* § 32.]

(1) Points P, P'; Q, Q'; ... in a st. line form a system in *involution* when $OP \cdot OP' = OQ \cdot OQ' = \ldots = OK^2 = OK'^2$. O is the *centre*; K, K' the *double-points* (*foci*), while P and P' are conjugate points.

(2) $\{PQRS\} = \{P'Q'R'S'\}$.

(3) $\{KPK'P'\} = -1$.

(4) (*a*) A system of points $P, Q, R, S, \ldots$ on one line is *homographic* with a system $P', Q', R', S', \ldots$ on another line when for *any four* $\{PQRS\} = \{P'Q'R'S'\}$.

(*b*) $PP', QQ', RR', \ldots$ all touch the same conic.

(*c*) Homographic systems on the same st. line are in involution.

AREAL COORDINATES

68. Cartesians into Areals.

$$\left.\begin{aligned} X &= l_1 x + m_1 y + n_1 z \\ Y &= l_2 x + m_2 y + n_2 z \end{aligned}\right\}, \text{ where } A \text{ is } (l_1, l_2),\ B \text{ is } (m_1, m_2),$$

C is (n_1, n_2).

Areals and Trilinear.

$$x = \frac{a\alpha}{2\Delta};\quad y = \frac{b\beta}{2\Delta};\quad z = \frac{c\gamma}{2\Delta}.$$

$$x + y + z = 1;\quad a\alpha + b\beta + c\gamma = 2\Delta.$$

69. Relations of Points.

(1) The point dividing the join of $(x_1, y_1, z_1)(x_2, y_2, z_2)$ in the ratio $n_1 : n_2$ is

$$x = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}, \quad y = \&c., \quad z = \&c.$$

(2) The distance of the two points is given by

$$-D^2 = a^2 (y_1 - y_2)(z_1 - z_2) + b^2 (z_1 - z_2)(x_1 - x_2) + c^2 (x_1 - x_2)(y_1 - y_2).$$

(3) *Area of Triangle* $(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \sin \omega.$$

(4) *Special Points,* Coordinates of:

Centroid : 1 : 1 : 1.

Circumcentre : $\sin 2A : \sin 2B : \sin 2C$.

Orthocentre : $\tan A : \tan B : \tan C$.

Incentre : $a : b : c$.

A-excentre : $-a : b : c$.

N.P. Centre : $\sin A \cos \overline{B-C} : \sin B \cos \overline{C-A} : \sin C \cos \overline{A-B}$.

Symmedian Point : $a^2 : b^2 : c^2$.

70. The Straight Line.

(1) *The Line Infinity* : $x + y + z = 0$.

The more correct form is $x+y+z = \mathrm{Lt}_{e=0}\, e(lx+my+nz)$. See *Askwith*, § 265.

(2) *General Equation* : $lx + my + nz = 0$.

Perpendicular Form : $px + qy + rz = 0$, where p, q, r are $\perp^{rs}$ from A, B, C on it.

To write a line in this form : *Divide throughout by*

$$\frac{1}{2\Delta} \sqrt{\text{sq. of line with } S\text{-substitution after squaring.}}$$

Note.—The phrase 'S-substitution' is used throughout this section to indicate the following :

For	x^2	y^2	z^2	yz ...
Write	a^2	b^2	c^2	$-bc \cos A$ ....

(3) *Join of* (x_1, y_1, z_1) (x_2, y_2, z_2) : $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$

St. line through (x_1, y_1, z_1) *parallel to* $lx + my + nz = 0.$

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ m-n & n-l & l-m \end{vmatrix} = 0.$$

Proof.—For it passes through the point $(lx+my+nz = 0,\ x+y+z = 0)$.

(4) *Condition of Parallelism of* $\left\{ \begin{matrix} l\,x + m\,y + n\,z = 0 \\ l'x + m'y + n'z = 0 \end{matrix} \right\}$.

$$P = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Condition of Perpendicularity :

S = (product of lines with S-substitutions after multiplication) = 0.

(5) *Angle between two lines*: $\tan\theta = 2\Delta P/S$. [See (4).]

(6) *Perpendicular Distance* from (X, Y, Z) to $px + qy + rz = 0$ [see (2)] is $pX + qY + rZ$.

71. Anharmonics.

(1) If the lines $l_1x + m_1y + n_1z = 0$, $l_2x + m_2y + n_2z = 0$, $l_3x + m_3y + n_3z = 0$, $l_4x + m_4y + n_4z = 0$ intersect, then their A.R. is $\dfrac{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \begin{vmatrix} l_3 & m_3 \\ l_4 & m_4 \end{vmatrix}}{\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \begin{vmatrix} l_4 & m_4 \\ l_1 & m_1 \end{vmatrix}}$.

(2) The A. R. of the pencil $u = 0$, $u + \mu v = 0$, $v = 0$, $u + \mu' v = 0$ is μ/μ' for any line coordinates whatever.

72. Choice of Triangle of Reference.

The Δ will usually be some triangle mentioned in the problem.

For N. P. C. the Δ formed by midpoints of sides or the pedal triangle is often convenient.

For a complete 4^1 the Δ formed by the three diagonals, in which case the sides may be written $\pm lx \pm my \pm nz = 0$, or the vertices taken as $(\pm f, \pm g, \pm h)$.

For Δ's $A_1B_1C_1$, $A_2B_2C_2$, ... in perspective, take O the centre of perspective at the centroid of the Δ of reference and let the vertices of this Δ lie on OA_1, OB_1, OC_1, so that OA_1 is $y = z$, &c., then the equations to the sides of any perspective Δ $A_1B_1C_1$ can be taken as $y + z + w_1 = 0$, $z + x + w_1 = 0$, $x + y + w_1 = 0$.

Change of Triangle of Reference.

Let (a_1, a_2, a_3) (b_1, b_2, b_3) (c_1, c_2, c_3) be the coordinates of the vertices of the new triangle, then the old coordinates in terms of the new are given by

$$x = a_1X + b_1Y + c_1Z,$$
$$y = a_2X + b_2Y + c_2Z,$$
$$z = a_3X + b_3Y + c_3Z.$$

73. General Equation of Second Degree.

$\phi = (A, B, C, D, E, F)(x, y, z)^2 = 0.$

$I = \phi$ after the S-substitutions have been made.

$H =$ the discriminant $\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix}$.

$K =$ the discriminant 'bordered by the line infinity and zero' $= \begin{vmatrix} A & F & E & 1 \\ F & B & D & 1 \\ E & D & C & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$.

(1) *Tangent* at (X, Y, Z): $x\phi_x + y\phi_y + z\phi_z = 0$.

Polar of (X, Y, Z): $x\phi_x + y\phi_y + z\phi_z = 0$, *provided* ϕ *is of the second degree.*

Notes.—(1) Beware of applying this formula to find the polar of $a^2/x + b^2/y + c^2/z = 0$, or of $\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$. It will give the tangent but *not* the polar [see § 20, *Note* (1)].

(2) The polar form is also that for the *chord of contact* of tangents from (X, Y, Z).

(2) *Condition of Tangency* of $lx+my+nz=0$: Equate to zero the discriminant bordered by the coefficients of the line and zero.

(3) *Pair of Tangents* from (X, Y, Z):

$$4\phi(x, y, z)\,\phi(X, Y, Z) = (x\phi_x + y\phi_y + z\phi_z)^2.$$

(4) *Centre*: $\phi_x = \phi_y = \phi_z$ *provided* ϕ *is of the second degree.*

Note.—Beware of following the Cartesian analogy and writing $\phi_x = 0$.

(5) *Conjugate* of $lx+my+nz=0$ is $\begin{vmatrix} \phi_x & \phi_y & \phi_z \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} = 0.$

Conic with $lx+my+nz=0$, $l'x+m'y+n'z=0$ for conjugate diameters is

$$A(lx+my+nz)^2 + B(l'x+m'y+n'z)^2 = 1.$$

(6) *Asymptotes*: $\phi + \frac{H}{K}(x+y+z)^2 = 0.$

Co-asymptotic Conics: $\phi + k(x+y+z)^2 = 0.$

Conics with asymptotes ∥ to those of ϕ:

$$\phi + (x+y+z)(lx+my+nz) = 0.$$

(7) *Foci of Conic*: Express that two tangents from (X, Y, Z), a focus, satisfy the conditions of (8) (*b*) below.

Directrix: Polar of Focus. *Centre*, see (4).

Axes: For *equations* compare § 53 (4) and § 53 (3).

For *magnitude*: If a, b be the semi-axes

$$ab = \frac{2\Delta H}{(-K)^{\frac{3}{2}}},\quad a^2+b^2 = -\frac{IH}{K^2}.$$

(8) *Analysis.*

(*a*) *Two Straight Lines*: $H=0$.

(*b*) *Circle*: $\dfrac{\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)^2\phi}{a^2} = \dfrac{\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial x}\right)^2\phi}{b^2} = \dfrac{\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^2\phi}{c^2}.$

(*c*) *Parabola*: $K=0$.

(*d*) *Rectangular Hyperbola*: $I = 0$.

(*e*) *Hyperbola*: K positive.

(*f*) *Ellipse*: K negative.

74. CIRCLE.

(1) *Circumcircle*: $a^2/x + b^2/y + c^2/z = 0$. [See § 75.]

(2) *General Equation*: $a^2yz + b^2zx + c^2xy$

$$= (lx + my + nz)(x + y + z)$$

$$= (p^2x + q^2y + r^2z)(x + y + z),$$

where p, q, r are the tangents from A, B, C to the circle.

(3) *Tangent from* (X, Y, Z):

$$T^2 = (p^2X + q^2Y + r^2Z)(X + Y + Z) - a^2YZ - b^2ZX - c^2XY.$$

(4) *Circles concentric with* ϕ: $\phi + k(x + y + z)^2 = 0$.

(5) *Condition* ϕ *is circle*: See § 73 (8) (*b*).

(6) *Special Circles.*

Incircle: $\sqrt{(s-a)x} + \sqrt{(s-b)y} + \sqrt{(s-c)z} = 0$.

A-excircle: $\sqrt{-sx} + \sqrt{(s-c)y} + \sqrt{(s-b)z} = 0$.

N. P. C.: $\Sigma a^2/(y+z-x) = 0$.

Self-Polar Circle: $\Sigma x^2 \cot A = 0$. [Centre : the orthocentre.]

Notes.—(1) For Feuerbach's Theorem [*GEOM.* § 17 (5)] find [by § 74 (2), *pqr*-form] $\Sigma x/(b-c) = 0$ as R. A. of N. P. C. and incircle, and show that it touches incircle. The point of contact of the two is

$$(b-c)^2(s-a) : (c-a)^2(s-b) : (a-b)^2(s-c).$$

(2) The common R. A. of N. P. C., circumcircle, and self-polar circle is $\Sigma x \cot A = 0$.

75. CIRCUMCONICS.

(1) *Curve*: $l/x + m/y + n/z = 0$.

Note.—The form $lyz + mzx + nxy = 0$ *must* be used to find the polar or two tangents.

(2) *Chord* $(x_1, y_1, z_1)(x_2, y_2, z_2)$:

$$lx/x_1x_2 + my/y_1y_2 + nz/z_1z_2 = 0.$$

(3) *Tangent* (x_1, y_1, z_1): $lx/x_1^2 + my/y_1^2 + nz/z_1^2 = 0$.

(4) $px+qy+rz=0$ is tangent if $\sqrt{lp}+\sqrt{mq}+\sqrt{nr}=0$.

(5) *Analysis*: *Circle* if $l:m:n=a^2:b^2:c^2$.

Parabola if $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$.

R.H. if it passes through the orthocentre.

76. INCONICS.

(1) *Curve*: $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$.

Note.—There is an ambiguity before each root, and they cannot all be taken as +. If the conic touch the sides in *D*, *E*, *F*, different signs will be needed for the ambiguities for the parts of the conic from *E* to *F*, from *F* to *D*, and from *D* to *E*.

(2) $px+qy+rz=0$ is a tangent if $\frac{l}{p}+\frac{m}{q}+\frac{n}{r}=0$.

Note.—Hence curve is a parabola if $l+m+n=0$.

(3) *Centre*: $m+n:n+l:l+m$.

77. SELF-POLAR CONICS.

(1) *Curve*: $lx^2+my^2+nz^2=0$.

(2) *Tangent at or Polar of* (x', y', z'): $lxx'+myy'+nzz'=0$.

(3) *Any two Conics* may be referred to a common self-polar triangle [*G. CON.* § 31 (2)] and therefore written

$$lx^2+my^2+nz^2=0,\quad l'x^2+m'y^2+n'z^2=0.$$

(4) *Auxiliary Angle* may often be usefully employed.

$$\sqrt{l}\,.\,x=i\sqrt{n}\,.\,z\cos\phi,\quad \sqrt{m}\,.\,y=i\sqrt{n}\,.\,z\sin\phi.$$

(5) *Analysis*: *Circle*: $l:m:n=\cot A:\cot B:\cot C$.

Parabola: $1/l+1/m+1/n=0$.

R.H.: $la^2+mb^2+nc^2=0$.

78. DOUBLE-CONTACT CONICS.

(1) *Curve*: $x^2=kyz$.

(2) *Any point* may be taken as $k\mu:k\mu^2:1$.

(3) *Tangent at* μ: $2\mu x=y+k\mu^2 z$.

(4) *Pole of* λ, μ: $(\mu+\lambda)/2:\mu\lambda:1/k$.

(5) *Analysis*: *Parabola*: $k=4$.

R.H.: $k=-a^2/bc\cos A$.

79. Tangential Coordinates.

(1) The tangential coordinates of a line are the perpendiculars p, q, r from the vertices of the Δ upon the line.

(2) *Invariant Relation.*

$$p^2a^2 + q^2b^2 + r^2c^2 - 2qrbc\cos A - \ldots = 0,$$

or $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) = 4\Delta^2.$

(3) *Equations*: *Circumconic*: $\sqrt{lp} + \sqrt{mq} + \sqrt{nr} = 0.$

Inconic: $l/p + m/q + n/r = 0.$

[For *Polar Reciprocation* and *Projection* see *G. CON.* §§ 35–6. For *Inversion* see *GEOM.* § 33, and *DIFF. CALC.* § 21 (2).]

DIFFERENTIAL CALCULUS

1. Elementary Laws.

(1) $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$.

(2) $\dfrac{d(u \pm v)}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$.

(3) $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$. Learn in words:

'Diff^{l} product = First × diff^{l} sec. + Second × diff^{l} first.'

(4) $\dfrac{d\left(\frac{u}{v}\right)}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$. Learn in words:

'Diff^{l} fraction $= \dfrac{\text{DENR.} \times \text{diff}^{l}\ \text{num}^{r} - \text{NUMR.} \times \text{diff}^{l}\ \text{den}^{r}}{\text{DENR.}^2}$.'

(5) $\dfrac{du}{dx} = \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$.

(6) $\dfrac{du}{dx} = 1 \Big/ \dfrac{dx}{du}$.

2. Standard Forms.

Form.	*Diff^{l}.*	*Form.*	*Diff^{l}.*
c	0	$\sin x$	$\cos x$
x^n	nx^{n-1}	$\cos x$	$-\sin x$
a^x	$a^x \log_e a$	$\tan x$	$\sec^2 x$
e^x	e^x	$\cot x$	$-\operatorname{cosec}^2 x$
$\log x$	$1/x$	$\sin^{-1} x$	$1/\sqrt{1-x^2}$
		$\tan^{-1} x$	$1/(1+x^2)$

Notes.—(1) $\frac{d}{dx}\log_a x = \log_a e/x$.

(2) $\sin^{-1}x + \cos^{-1}x = \pi/2$. Hence $\frac{d}{dx}\sin^{-1}x = -\frac{d}{dx}\cos^{-1}x$.

So for $\cot^{-1}x$.

Standard Forms (for Reference).

Form.	*Diff^l.*	*Form.*	*Diff^l.*
$\sec x$	$\frac{\sin x}{\cos^2 x}$	$\sinh x$	$\cosh x$
$\operatorname{cosec} x$	$-\frac{\cos x}{\sin^2 x}$	$\cosh x$	$\sinh x$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$	$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{vers}^{-1} x$	$\frac{1}{\sqrt{2x-x^2}}$	$\coth x$	$-\operatorname{cosech}^2 x$
		$\operatorname{sech} x$	$-\frac{\sinh x}{\cosh^2 x}$
		$\operatorname{cosech} x$	$-\frac{\cosh x}{\sinh^2 x}$

Form.	*Equivalent.*	*Diff^l.*
$\sinh^{-1} x$	$= \log(x + \sqrt{1+x^2})$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$= \log(x + \sqrt{x^2-1})$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$= \frac{1}{2}\log\frac{1+x}{1-x}$	$\frac{1}{1-x^2}(x<1)$
$\coth^{-1} x$	$= \tanh^{-1}\frac{1}{x}$	$-\frac{1}{x^2-1}(x>1)$
$\operatorname{sech}^{-1} x$	$= \cosh^{-1}\frac{1}{x}$	$-\frac{1}{x\sqrt{1-x^2}}$
$\operatorname{cosech}^{-1} x$	$= \sinh^{-1}\frac{1}{x}$	$-\frac{1}{x\sqrt{x^2+1}}$

3. Methods of Differentiation.

(1) u^v, uvw, &c. Take logs.

(2) Note such transformations as:

$$\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x; \tan^{-1}\frac{1+x}{1-x} = \frac{\pi}{4} + \tan^{-1}x.$$

4. Partial Fractions.

(1) If the numerator is of equal or higher degree than the denominator divide out by the latter.

(2) The partial fractions for $f(x)/(x-a)^r\,(x^2+bx+c)^s$ are

$$\frac{A_1}{(x-a)^r} + \frac{A_2}{(x-a)^{r-1}} + \ldots + \frac{A_r}{x-a} + \frac{B_1x+C_1}{(x^2+bx+c)^s} + \ldots + \frac{B_sx+C_s}{x^2+bx+c},$$

and the values of A, B, C's may always be obtained by multiplying up and equating coefficients; but this is *never* the best method, though one or more of the coefficients may usually be obtained this way easily.

(3) *Unrepeated Linear Factors.* $\dfrac{f(x)}{F(x)} = \Sigma\dfrac{f(a)}{F'(a)}\cdot\dfrac{1}{x-a}$.

Where $f(x)$ is of lower degree than $F(x)$,

$$F(x) = (x-a_1)\,(x-a_2)\ldots(x-a_n).$$

This rule may be expressed thus:

'If $F(x)$ contains a factor $x-a$ the P. F. corresponding to $x-a$ is $A/(x-a)$, where A is the result of putting $x=a$ in every part of the original fraction except $x-a$ itself.'

Notes.—(1) If no factors are repeated this at once gives the expression of the fraction.

Ex. $\dfrac{x-c}{(x-a)\,(x-b)} = \dfrac{a-c}{a-b}\cdot\dfrac{1}{x-a} + \dfrac{b-c}{b-a}\cdot\dfrac{1}{x-b}$.

(2) The text-books usually give the rule as applicable only to unrepeated factors. As a fact, however, it is true if for $x-a$ we read $(x-a)^r$, and gives the coefft. for the *highest* power of the repeated factor.

Ex. $\dfrac{x-3}{(x-1)^2(x-2)} = \dfrac{1-3}{1-2}\cdot\dfrac{1}{(x-1)^2} + \dfrac{2-3}{(2-1^{\,2}}\cdot\dfrac{1}{x-2} + \dfrac{A}{x-1}$, and A can be determined by (2) above. (*Ans.* $A = 1$.)

(3) And this method is the quickest if no factor is repeated more than twice.

(4) *Repeated Linear Factors.*

Fraction $= f(x)/(x-a)^m(x-b)^n F(x)$.

Put $x = a+h$ in the terms which do not contain $(x-a)$ and expand $f(a+h)/(a-b+h)^n F(a+h)$ in ascending powers of h. *Contract the division so as to give the first m coefficients only,* $A_1, A_2, \ldots A_m$, then the P. F.'s corresponding to $(x-a)^m$ are

$$\frac{A_1}{(x-a)^m} + \frac{A_2}{(x-a)^{m-1}} + \ldots + \frac{A_m}{x-a}.$$

Then put $x = b+h$ and work similarly for the terms corresponding to $(x-b)^n$.

Note.—Application of (2) will often now complete the decomposition.

(5) *Repeated Non-Linear Factors.*

Ex. $\dfrac{x^3}{(x^2+1)^2(x^2-x+1)} = \dfrac{ax+b}{(x^2+1)^2} + \dfrac{cx+d}{x^2+1} + \dfrac{ex+f}{x^2-x+1}$.

Hence x^3

$= (ax+b)(x^2-x+1) + (cx+d)(x^2+1)(x^2-x+1) + (ex+f)(x^2+1)^2$.

Put $x^2 = -1$, $x^3 = -x$ and reduce to $-x = a-bx$; $\therefore a = 0, b = 1$.

Put $x^2 = x-1$, $x^3 = -1$, &c., and prove similarly $e = 1, f = 0$.

Use (2) to deduce $c = d = -1$.

But for practical applications (differential and integral calculus) it is usually necessary to employ the complex linear factors of these non-linear expressions.

5. Successive Differentiation.

(1) $D^r x^n = n(n-1)\ldots(n-r+1)\,x^{n-r}$.

(2) $D^n\left[\dfrac{f(x)}{F(x)}\right]$. Use partial fractions if $f(x)$, $F(x)$ are integ. algeb. functions.

(3) $D^n\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{\sin}{\cos}\left(x + \dfrac{n\pi}{2}\right)$.

(4) $D^2 = -1$ when operating on $\begin{matrix}\sin\\ \cos\end{matrix} x$; $D^2 = -m^2$ operating on $\begin{matrix}\sin\\ \cos\end{matrix} mx$; $f(D^2)\begin{matrix}\sin\\ \cos\end{matrix} mx = f(-m^2)\begin{matrix}\sin\\ \cos\end{matrix} mx$.

(5) *Leibnitz's Theorem.*

$$D^n(uv) = (D_1 + D_2)^n uv \qquad \begin{bmatrix} D_1 \text{ operating on } u \text{ only} \\ D_2 \quad ,, \quad ,, \ v \quad ,, \end{bmatrix}$$

$$= v\frac{d^n u}{dx^n} + n\frac{dv}{dx}\cdot\frac{d^{n-1}u}{dx^{n-1}} + \ldots.$$

Proof given of this should be by induction. In giving this proof take care to deal with the general term.

Note.—$D_1 = D - D_2$; $\therefore D_1^n = (D - D_2)^n$;

$$\therefore v\frac{d^n u}{dx^n} = \frac{d^n(uv)}{dx^n} - n\frac{d^{n-1}(uv)}{dx^{n-1}}\cdot\frac{dv}{dx} + \ldots.$$

(6) $D^n[e^{ax}.X] = e^{ax}.(D+a)^n X$, and, in general,
$f(D)[e^{ax}.X] = e^{ax}.f(D+a)X$.

(7) $\sin^m x \cos^n x$. Turn into multiple angles.

$\tan^{-1}\dfrac{ax+b}{c}$. After first diff^n split into P. F.'s and use De Moivre.

$x^r \sin ax$, $x^r \sin^m x \cos^n x$. Use Leibnitz.

(8) $[D^n y]_{x=0}$ can often be found by forming a diff^l equation.

Exx. $y = [\sin^{-1}x]^2$, $[\sinh^{-1}x]^2$, $\sin(m\sin^{-1}x)$, $e^{m\tan^{-1}x}$, &c.

6. Expansions.

(1) When $f(x)$ changes sign with x [e. g. $\sin x$] only odd powers of x occur in its expansion; if it does not change sign, only even powers.

(2) If the expansion of $\dfrac{df(x)}{dx}$ or of $\dfrac{d^2 f(x)}{dx^2}$ is known [e. g. $f(x) = e^x$, $\sin x$, $\sin^{-1}x$] we may obtain expansion of $f(x)$

by assuming $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and equating coefficients in

$$\frac{df(x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \text{Known Series.}$$

(3) *Taylor's Theorem.*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^n}{n!} f^n(x+\theta h).$$

Remainder in Taylor's Theorem.

Lagrange's Form: $\dfrac{h^n}{n!} f^n(x+\theta h)$. [$\theta$ a prop. frac.

Cauchy's Form: $\dfrac{(1-\theta)^{n-1}.h^n}{(n-1)!} f^n(x+\theta h)$.

Schlömilch's Form: $\dfrac{(1-\theta)^{n-p-1}.h^n}{(n-1)!.(p+1)} f^n(x+\theta h)$.

$$\begin{cases} = \text{Cauchy, } p = 0, \\ = \text{Lagrange, } p = n-1. \end{cases}$$

Proof.—Let $\phi(x+h) = \phi(x) + h\phi'(x) + \ldots + \dfrac{h^{n-1}}{(n-1)!} \phi^{n-1}(x) + \dfrac{h^n}{n!} R.$

Consider the function $u = \phi(x+h) - \phi(y) - (x+h-y)\,\phi'(y) - \ldots$

$$- \frac{(x+h-y)^{n-1}}{(n-1)!} \phi^{n-1}(y) - \frac{(x+h-y)^n}{n!} R.$$

Then u is a function of y which $= 0$ if $y = x$, and $= 0$ if $y = x+h$; $\therefore$ $du/dy = 0$ for some value of y between x and $x+h$, say $x+\theta h$; $\therefore$ $R = \phi^n(x+\theta h)$. For *Cauchy's Remainder* take hR for the last term of $\phi(x+h)$. For *Schlömilch's Remainder* take for the last term

$$h^{p+1} R/(p+1).$$

(4) *Maclaurin's Theorem.*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(\theta x).$$

(5) *Failure of Taylor's Theorem.* Taylor's Theorem fails if $f(x)$ or one of its differential coeffts. becomes infinite or discontinuous between the values considered, or if

$$\text{Lt}_{n=\infty} \frac{h^n}{n!} f^n(x+\theta h) \neq 0.$$

(6) *Expansion by a Differential Equation.* Whenever $f^n(x)$ can be found in a finite form Maclaurin's Theorem may be applied directly. In other cases $f^n(0)$ can often be found by a diff[1] equation [cf. § 5 (8)] connecting $f^n(0)$ with $f^{n-1}(0)$ or $f^{n-2}(0)$ and then Maclaurin's Theorem applied.

Exx. $(\sin^{-1}x)^2$, $\sin(m\sin^{-1}x)$, $(\tan^{-1}x)^2$, $e^{a\sin^{-1}x}$, $\dfrac{\sinh^{-1}x}{\sqrt{1+x^2}}$, &c.

(7) *Arbogast's Theorem.*

$$\phi\left(a_0 + a_1x + a_2\frac{x^2}{2!} + \ldots\right) = \phi(a_0) + x\Delta\phi(a_0) + \frac{x^2}{2!}\Delta\,.\,\Delta\phi(a_0) + \ldots,$$

where $$\Delta = a_1\frac{\partial}{\partial a_0} + a_2\frac{\partial}{\partial a_1} + \ldots.$$

Exx. $\sin\left(a+bx+\dfrac{cx^2}{2}\right) = \sin a + b\cos a\,.\,x$

$$+ (c\cos a - b^2\sin a)\,x^2/2\,! + (-3bc\sin a - b^3\cos a)\,x^3/3\,! + \ldots,$$

$$\log(a+bx+\tfrac{1}{2}cx^2) = \log a + \frac{b}{a}x + \left(\frac{c}{a} - \frac{b^2}{a^2}\right)\frac{x^2}{2!} + \left(-\frac{3bc}{a^2} - \frac{2b^3}{a^3}\right)\frac{x^3}{3!} + \ldots.$$

(8) *Lagrange's Theorem.* If $x = y + a\phi(x)$,

$$F(x) = F(y) + a\phi(y)F'(y) + \frac{a^2}{2!}\frac{d}{dy}\left[\{\phi(y)\}^2F'(y)\right] + \ldots + \frac{a^n}{n!}\frac{d^{n-1}}{dy^{n-1}}\left[\{\phi(y)\}^nF'(y)\right] + \ldots.$$

Ex. Let $x = ae^x$. Here $y = 0$, $\phi(x) = e^x$. Let $F(x) = x$, $F'(x) = 1$,

$$\therefore\ x = a + \frac{a^2}{2!}\left[\frac{d}{dy}\,.\,e^{2y}\right]_{y=0} + \ldots + \frac{a^n}{n!}\left[\frac{d^{n-1}}{dy^{n-1}}\,.\,e^{ny}\right]_{y=0} + \ldots$$

$$= a + 2a^2/2\,! + \ldots + n^{n-1}\,.\,a^n/n\,! + \ldots,$$

a series which is convergent if $a < 1/e$.

Hence, e.g., the solution of $10x = e^x$ is $x = \cdot111832\ldots$.

(9) *Laplace's Theorem.* If $x = f\{y + a\phi(x)\}$,

$$F(x) = Ff(y) + a\phi f(y)\frac{dFf(y)}{dy} + \frac{a^2}{2}\frac{d}{dy}\left[\{\phi f(y)\}^2\frac{dFf(y)}{dy}\right] + \ldots.$$

(10) *Burmann's Theorem.* The object of this theorem is to expand any function $F(x)$ in powers of any other function $f(x)$. The function $f(x)$ is written in the form $\frac{x-y}{\phi(x)}$, where y is a constant which may or may not be zero. If we put $f(x) = \frac{x-y}{\phi(x)} = a$, the required problem is now to express $F(x)$ in powers of a. This is effected by Lagrange.

The name is sometimes applied to the particular case: If $\phi^{-1}(x)$ be the inverse function of $\phi(x)$ so that $\phi\phi^{-1}(x) = x$, and if $\phi(0) = 0$,

$$\phi^{-1}(x) = x\left[\frac{x}{\phi(x)}\right]_{x=0} + \frac{x^2}{2!}\left[\frac{d}{dx}\cdot\frac{x^2}{[\phi(x)]^2}\right]_0 + \frac{x^3}{3!}\left[\frac{d^2}{dx^2}\cdot\frac{x^3}{[\phi(x)]^3}\right]_0 + \dots.$$

Note.—In order that the answer may have a simple form $f(x)$ must be such that it may be written as $(x-y)/\phi(x)$ where $\phi(x)$ is simple.

Exx. $f(x) = xe^{kx} = (x-0)/e^{-kx}$; $\therefore\ y = 0,\ \phi(x) = e^{-kx}$.

$f(x) = (x-1)/(x^3+1)$; $\therefore\ y = 1,\ \phi(x) = x^3+1$.

7. Bernoulli's Numbers.

(1) The most convenient treatment of these assumes

$$\tan x = S_1\cdot\frac{x}{1!} + S_3\cdot\frac{x^3}{3!} + S_5\cdot\frac{x^5}{5!} + \dots,$$

$$\sec x = \quad 1 + S_2\cdot\frac{x^2}{2!} + S_4\cdot\frac{x^4}{4!} + \dots,$$

where S_1, S_3, ... are 'prepared Bernoullians', and S_2, S_4, ... are Euler Numbers.

(2) *To evaluate the numbers*:

$$y = \sec x + \tan x = 1 + S_1\cdot\frac{x}{1!} + S_2\cdot\frac{x^2}{2!} + \dots,$$

$$\therefore\ y\cos x = 1 + \sin x.$$

Differentiate n times and put $x = 0$,

$$\therefore\ S_n - \frac{n(n-1)}{2!}S_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4!}S_{n-4} - \dots + \cos\frac{n\pi}{2} = \sin\frac{n\pi}{2}.$$

(3) $S_1 = 1$, $S_2 = 1$, $S_3 = 2$, $S_4 = 5$, $S_5 = 16$, $S_6 = 61$, $S_7 = 272$, $S_8 = 1385$, $S_9 = 7936$, $S_{10} = 50521$, &c.

(4) The relation connecting S_{2n-1} with the ordinary Bernoulli Number is

$$S_{2n-1} = B_{2n-1}\frac{2^{2n}(2^{2n}-1)}{2n}.$$

(5) The Bernoulli Numbers are : $B_2 = B_4 = B_6 = \ldots = 0$, $B_0 = \frac{1}{2}$, $B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$, $B_7 = \frac{1}{30}$, $B_9 = \frac{5}{66}$, $B_{11} = \frac{691}{2730}$, $B_{13} = \frac{7}{6}$, $B_{15} = \frac{3617}{510}$, $B_{17} = \frac{43867}{798}$, &c.

After B_5 they continually increase to a limit ∞. There is unfortunately divergence among mathematicians as to the notation. What is given above as B_{2n-1} (following Edwards) is often (Chrystal) taken as B_n, sometimes as B_{2n}, while German writers do not take them all as $+$.

(6) The following results are added for reference :

(a) $$1+\left(-\frac{1}{3}\right)^n+\left(\frac{1}{5}\right)^n+\left(-\frac{1}{7}\right)^n+\ldots = \frac{1}{2}\left(\frac{\pi}{2}\right)^n \cdot \frac{S_{n-1}}{(n-1)!}.$$

Hence the sum of $1+(1/2)^{2n}+(1/3)^{2n}+(1/4)^{2n}+\ldots$.

(b) $$\frac{1}{2}\cdot\frac{e^x+1}{e^x-1} = \frac{1}{x}+\frac{B_1 x}{2!}-\frac{B_3 x^3}{4!}+\frac{B_5 x^5}{6!}-\ldots,$$

$$\coth x = \frac{1}{x}+\frac{2^2}{2!}B_1 x-\frac{2^4}{4!}B_3 x^3+\ldots,$$

$$\cot x = \frac{1}{x}-\frac{2^2}{2!}B_1 x-\frac{2^4}{4!}B_3 x^3+\ldots.$$

$$\operatorname{cosec} x = \frac{1}{x}+\frac{2(2-1)}{2!}B_1 x+\frac{2(2^3-1)}{4!}B_3 x^3+\ldots,$$

$$\tan x = \frac{2^2(2^2-1)}{2!}B_1 x+\frac{2^4(2^4-1)}{4!}B_3 x^3+\ldots,$$

$$\tanh x = \frac{2^2(2^2-1)}{2!}B_1 x-\frac{2^4(2^4-1)}{4!}B_3 x^3+\ldots.$$

(c) The following may also be expanded in Bernoullians :

$\log(\sin x/x)$, $\log(\sinh x/x)$, $\log\cos x$, $\log\cosh x$, $\operatorname{cosec}^2 x$, &c.

(d) $$1^r+2^r+3^r+\ldots+n^r = \frac{1}{r+1}[(n+B)^{r+1}-B^{r+1}],$$

$$(1+B)^n = n+B^n$$

where after expansion B_0 is written for B^1 and $i^{n-2}B_{n-1}$ for B^n.

8. Partial Differentiation.

(1) If $u = \phi(x_1, x_2, \ldots)$ where x_1, x_2, ... are functions of x, or of some other variable,

$$du = \frac{\partial \phi}{\partial x_1} \cdot dx_1 + \frac{\partial \phi}{\partial x_2} \cdot dx_2 + \ldots.$$

Note.—If $\phi(x, y) = 0$ and ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yy} be p, q, r, s, t,

$$\frac{dy}{dx} = -\frac{p}{q}; \quad \frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

(2) $$\frac{\partial^2 \phi}{\partial x \,.\, \partial y} = \frac{\partial^2 \phi}{\partial y \,.\, \partial x}.$$

(3) *Extension of Taylor's Theorem.*

$$\phi(x+h, y+k) = e^{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}} \phi(x, y)$$

$$= \phi(x, y) + \left(h\frac{\partial \phi}{\partial x} + k\frac{\partial \phi}{\partial y}\right) + \frac{h^2\frac{\partial^2 \phi}{\partial x^2} + 2hk\frac{\partial^2 \phi}{\partial x \partial y} + k^2\frac{\partial^2 \phi}{\partial y^2}}{2!} + \ldots.$$

So for three or more variables.

(4) If u, v be functions of x, y and $u_x v_y - u_y v_x = 0$, then u will be a function of v. And conversely.

(5) *Euler's Theorems.* If u be a homogeneous n-tic,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u, \text{ \&c.}$$

9. Change of the Independent Variable.

(1) To change from x, y to u, v

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial v}.$$

To find $\frac{\partial u}{\partial x}$ use an equation containing u, x, y only.

(2) In this work NEVER use $\frac{\partial u}{\partial x} = 1 \Big/ \frac{\partial x}{\partial u}$, for this equation involves the assumption that a relation exists between u and x which does not involve any other variable.

For reference only:

(3) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, from which, by putting $t = y$, $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$.

(4) $\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)^3$, $\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2} \Big/ \left(\frac{dx}{dy}\right)^3$.

(5) *Polars to Cartesians*:

$$\frac{\partial}{\partial x} = \cos\theta \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial}$$

$$\frac{\partial}{\partial y} = \sin\theta \cdot \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial\theta}.$$

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

$$\frac{\partial}{\partial\theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial\theta^2}.$$

10. OPERATORS.

(1) $\theta = x\frac{d}{dx}$. $6^n \cdot x^a = a^n \cdot x^a$.

$$\phi(\theta) \cdot (x^n u) = x^n \phi(\theta + n) u.$$

$$x^n \frac{d^n}{dx^n} = \theta(\theta - 1)(\theta - 2) \ldots (\theta - n + 1).$$

(2) $Eu_x = u_{x+1}$, $\Delta u_x = u_{x+1} - u_x$, $Du_x = \frac{du_x}{dx}$,

$$E = 1 + \Delta = e^D.$$

(3) *Taylor's Theorem.*

$$\phi(x + h) = E^h \phi(x) = e^{hD}\phi(x) = \ldots.$$

$$\phi(x + h, y + k) = E^h \cdot E^k \phi(x, y) = e^{hD + kD'} \phi(x, y)$$

$$= \left[1 + (hD + kD') + \frac{(hD + kD')^2}{2!} + \ldots\right] \phi(x, y) = \ldots.$$

11. Undetermined Forms.

(1) *Algebraical Methods.* Suppose the form to become 'undetermined' when $x = a$, put $x = a + h$ and expand everything in powers of h.

(2) *Differential Methods.*

(*a*) $\frac{0}{0}$. If $\left[\frac{\phi(x)}{\psi(x)}\right]_{x=a} = \frac{0}{0}$ its value is $\frac{\phi'(a)}{\psi'(a)}$. If this is of the form $\frac{0}{0}$ the value $= \frac{\phi''(a)}{\psi''(a)}$ and so on.

(*b*) $\frac{\infty}{\infty}$. If $\frac{\phi}{\psi} = \frac{\infty}{\infty}$ its value is $\frac{\phi'}{\psi'}$. This is *necessarily* also of form $\frac{\infty}{\infty}$, but the infinite factor will often cancel.

(*c*) $0 \times \infty$. If $\phi \times \psi = 0 \times \infty$ write it $\frac{\phi}{\frac{1}{\psi}} = \frac{0}{0}$.

(*d*) $\infty - \infty = \phi - \psi = \psi\left(\frac{\phi}{\psi} - 1\right)$. This is of form $\infty \times 0$.

Or thus, $e^{\phi-\psi} = e^{\phi}/e^{\psi}$. This is of form $\frac{\infty}{\infty}$.

(*e*) 0^0, ∞^0, 1^∞. Evaluate the logarithms.

12. Maxima and Minima (Elementary Methods). See *ALG.* §§ 8 (5), 27; *TRIG.* § 16; *GEOM.* §§ 30, 39 (7) (8).

13. Maxima and Minima—Single Variable.

(1) *Rule*: To find the max. or min. values of $\phi(x)$, find the values of x for which $\frac{d\phi}{dx} = 0$ or ∞. *For values making* $\frac{d\phi}{dx} = 0$ we have $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ if $\frac{d\phi}{dx}$ changes from $\begin{matrix}+ \text{ to } -\\ - \text{ to } +\end{matrix}$, i.e. if $\frac{d^2\phi}{dx^2}$ is $\begin{matrix}-\\ +\end{matrix}$. *For values making* $\frac{d\phi}{dx} = \infty$ we *may* have a $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ under the same conditions and *shall* have such a value if x passes THROUGH the critical value as we travel along the function.

If $\frac{d^2\phi}{dx^2} = 0$, then for $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ we must have also $\frac{d^3\phi}{dx^3} = 0$, the first diff[1] coefft. which does not vanish must be of an even order and must fulfil the sign-conditions given above.

(2) Between two equal values of a function at least one max. or min. value must occur.

(3) Maxima and minima values occur alternately.

(4) If y and x are connected by an implicit relation $\phi(x, y) = 0$, the formulae of 8 (1), *Note*, must be used.

14. (1) MAXIMA AND MINIMA: TWO INDEPENDENT VARIABLES. $w = \phi(x, y)$.

(*a*) For a $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ value of w we must have $\frac{\partial\phi}{\partial x} = 0$ and also $\frac{\partial\phi}{\partial y} = 0$.

(*b*) If $r = \phi_{xx}$, $s = \phi_{xy}$, $t = \phi_{yy}$, then unless r, s, t all vanish when

$rt < s^2$ we have neither max. nor min. value for w;

$rt > s^2$ a $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ value as r (or t) is $\begin{matrix}-\\ +\end{matrix}$;

$rt = s^2$, the case is indeterminate and higher diff[1] coeffts. must be examined.

(2) MAXIMA AND MINIMA: THREE INDEPENDENT VARIABLES. $w = \phi(x, y, z)$.

(*a*) For a $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ value of w we must have

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial z} = 0.$$

(*b*) If A, B, C, F, G, H represent ϕ_{xx} ..., ϕ_{yz} ..., we have

Minimum value of w if A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$	are	+	+	+
Maximum	,,	,,	,,	,,	,,	−	+	−
Neither min. max.	,,	,,	,,	,,	have any other signs.			

15. MAXIMA AND MINIMA: SEVERAL DEPENDENT VARIABLES.

To find the $\begin{matrix}\text{max.}\\ \text{min.}\end{matrix}$ values of $w = \phi(x, y, z)$ subject to

$$\left.\begin{matrix} f(x, y, z) = 0 \\ F(x, y, z) = 0 \end{matrix}\right\}.$$

$$dw = 0 = \phi_x \cdot dx + \phi_y \cdot dy + \phi_z \cdot dz,$$
$$0 = f_x \cdot dx + f_y \cdot dy + f_z \cdot dz,$$
$$0 = F_x \cdot dx + F_y \cdot dy + F_z \cdot dz.$$

Eliminate λ and μ from $\left.\begin{matrix} \phi_x + \lambda f_x + \mu F_x = 0 \\ \phi_y + \lambda f_y + \mu F_y = 0 \\ \phi_z + \lambda f_z + \mu F_z = 0 \end{matrix}\right\}$.

PROPERTIES OF CURVES

16. TANGENT.

(1) **Curve**: $y = f(x)$. **Tangent**: $Y - y = \dfrac{dy}{dx}(X - x)$.

(2) **Curve**: $f(x, y) = 0$.

Tangent: $(X - x)\dfrac{\partial f}{\partial x} + (Y - y)\dfrac{\partial f}{\partial y} = 0$.

(3) **Curve**: $f(x, y, z) = 0$, where $z = 1$ renders curve homogeneous. **Tangent**: $X\dfrac{\partial f}{\partial x} + Y\dfrac{\partial f}{\partial y} + Z\dfrac{\partial f}{\partial z} = 0$.

Note.—If (x, y) be used as current coordinates this equation gives the 'Polar Curve' of (X, Y) with regard to $f(x, y) = 0$.

(4) At origin, if this be on the curve: Equate to zero the terms of lowest order in x, y.

(5) $n(n-1)$ tangents can be drawn from an external point to a curve of nth degree.

(6) *Polar Equation.* The tangent at (U, α) is

$$u = U\cos(\theta-\alpha)+U'\sin(\theta-\alpha), \text{ where } U' = \left(\frac{du}{d\theta}\right)_{\theta=\alpha}.$$

Proof.—Assume it to be $u = A\cos(\theta-\alpha)+B\sin(\theta-\alpha)$ and express that u and $\frac{du}{d\theta}$ are the same for the curve and the tangent when $\theta = \alpha$.

17. Normal.

(1) **Curve:** $y = f(x)$. **Normal:** $(Y-y)\frac{dy}{dx}+(X-x)=0.$

(2) **Curve:** $f(x, y) = 0$. **Normal:** $\frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}}.$

(3) n^2 normals can be drawn from an external point to a curve of the nth degree.

(4) *Polar Equation.* The normal at (U, α) is

$$\frac{U'}{U}\cdot u = U'\cos(\theta-\alpha)-U\sin(\theta-\alpha).$$

18. Related Lines. The figure should be committed to memory with the various lines associated with it. The following properties follow from it and should not be learnt individually, but deduced from it as required, until the habit of doing so becomes fixed.

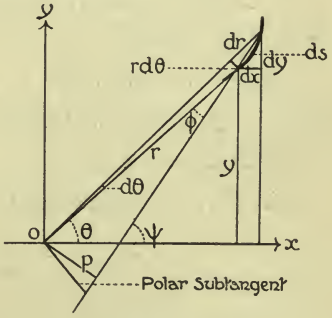


Fig. 47.

$ds^2 = dx^2 + dy^2 = dr^2 + (rd\theta)^2.$

$$\cos\phi = \frac{dr}{ds},\ \sin\phi = \frac{rd\theta}{ds},$$

$$\tan\phi = \frac{rd\theta}{dr}.$$

$$\cos\psi = \frac{dx}{ds},\ \sin\psi = \frac{dy}{ds},$$

$$\tan\psi = \frac{dy}{dx}.$$

$$OT = x - y\frac{dx}{dy}.\quad p = r\sin\phi = \frac{r^2 d\theta}{ds}.$$

LEARN only *Polar Subtangent*: OZ (Fig. 47) $= r^2 \dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$.

Perpendicular: $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2 = u^2 + \left(\dfrac{du}{d\theta}\right)^2$.

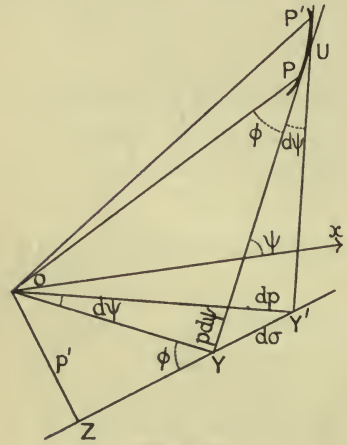


FIG. 48.

19. THE PEDAL FIGURE. Here again the figure should be learnt.

Locus of $\dfrac{Y}{Z}$ is the $\begin{matrix}\text{first}\\ \text{second}\end{matrix}$ pedal.

$OYY'U$ concyclic, ϕ of curve $= \phi$ of pedal.

$PY = \dfrac{dp}{d\psi}$, $p' = p \sin\phi = \dfrac{p^2}{r}$

$d\sigma^2 = dp^2 + (pd\psi)^2$. None of these are to be remembered.

20. FORMS OF EQUATION TO CURVE.

(1) **Pedal Form** (i.e. p-r equation; not to be confused with Pedal Curve).

(*a*) *Polar Coordinates.*

$$p = r\sin\phi, \quad \tan\phi = \frac{rd\theta}{dr}, \quad f(r, \theta) = 0.$$

Eliminate ϕ and θ.

(*b*) *Cartesian Coordinates.* Since the tangent is

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0,$$

$$p^2 = \frac{\left(\dfrac{\partial f}{\partial z}\right)^2}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}, \quad r^2 = x^2 + y^2, \quad f(x, y) = 0.$$

Eliminate x and y.

Common Curves (*Pedal Equations*).

St. Line : $p = a$.

Circle (centre) : $p = a$, or $r = a$, or $p = r$.

Circle (circumf.) : $2ap = r^2$.

Circle (any origin) : $2ap = r^2 - b^2$.

**Parabola* (focus) : $p^2 = ar$.

**Ellipse* / **Hyperbola* (focus) : $\frac{b^2}{p^2} = \frac{2a}{r} \mp 1$.

**Eq. Spiral* : $p = ar$.

**Epicycloid* / **Hypocycloid* $p^2 = ar^2 + b$.

Circle Involute : $r^2 = p^2 + a^2$.

Cardioid (cusp) : $r^3 = ap^2$. Also of epicycloidal form [§ 28 (6)].

Lemniscate (node) : $r^3 = a^2 p$.

Cotes' Spirals: $1/p^2 = a/r^2 + b$ [§ 28].

An asterisk indicates the curves in which the p-r equation should be learnt.

(2) **Tangential-Polar** (i. e. p-ψ equation).

Find the pedal curve [see § 21 (1)]. Let it be $r = f(\theta)$.

Then we have $\frac{\pi}{2} - \psi = -\theta$ and $p = r$, $\therefore\ p = f\left(\psi - \frac{\pi}{2}\right)$.

Common Curves (p, ψ form).

Point : $p = a \sin \psi$.

St. Line : $\psi = \alpha$.

Circle (circumf.) : $p = a(1 + \sin \psi)$.

Parabola (focus) : $p = a \operatorname{cosec} \psi$.

Parabola (vertex) : $p = a \operatorname{cosec} \psi \,.\, \cos^2 \psi$.

Rect. Hyp. (cent.) : $p^2 = -a^2 \cos 2\psi$.

**Ellipse* (cent.) : $p^2 = a^2 \sin^2 \psi + b^2 \cos^2 \psi$.

(3) **Intrinsic Equation** (i. e. s-ψ equation) can only rarely be determined without the Integral Calculus. Eliminate x between

$$\left\{ \begin{aligned} \tan \psi &= f'(x) \\ s &= \int \sqrt{1 + [f'(x)]^2}\, dx \end{aligned} \right\}.$$

Common Curves (Intrinsic).

Circle : $s = a\psi$.

Catenary : $s = c \tan \psi$.

Cycloid : $s = 4a \sin \psi$.

Epi-(Hypo-)cycloid : $s = a \sin b\psi$.

Circle Involute : $s = \frac{1}{2} a \psi^2$.

Parabola Evolute : $s = 2a(\sec^3 \psi - 1)$.

(4) **Tangential Equation** (i. e. condition that $lx + my + n = 0$ touches $F(x, y) = 0$).

(*a*) Compare

$$XF_x + YF_y + ZF_z = 0 \text{ and } lX + mY + n = 0;$$

$$\therefore F_x/l = F_y/m = F_z/n = \lambda.$$

But $F(x, y) = 0$. Eliminate X, Y, λ.

(*b*) Make $F(x, y) = 0$ homogeneous by means of

$$lx + my + n = 0,$$

and express condition that resulting eqn. has two equal roots.

21. Related Curves.

(1) **Pedal Curves**: *To find the Pedal Curve.*

(*a*) *p, r-Form.* If $f(r, p) = 0$ be the original curve, $p' = p^2/r$, $r' = p$; $\therefore$ the pedal curve is $f(r^2/p, r) = 0$.

(*b*) *Polars.* *Method* i.

$$\begin{cases} \theta = \theta' + \dfrac{\pi}{2} - \phi. & \text{[Draw init. line so that } \theta' \text{ is positive.]} \\ \tan\phi = r\dfrac{d\theta}{dr}. \\ r' = r\sin\phi. \\ \dfrac{1}{r'^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2. \end{cases}$$

Eliminate r, θ, ϕ.

Method ii. Find the envelope of circles described on the radii vectores as diameters.

(*c*) *Cartesians.* The condition that $X\cos\alpha + Y\sin\alpha = p$ should touch $F(x, y) = 0$ is

$$\frac{\partial F}{\partial x}\Big/\cos\alpha = \frac{\partial F}{\partial y}\Big/\sin\alpha = \frac{\partial F}{\partial z}\Big/(-p).$$

Eliminating x, y we have an equation in p, α; i.e. the polar coords. of point Y.

(*d*) The first negative pedal is the envelope of a st. line drawn through any point of the curve at right angles to the radius vector.

(*c*) *Common Pedals* :

Curve.	*Origin.*	*Pedal.*
Circle.	On circumference.	Cardioid.
,,	Anywhere.	Limaçon.
Parabola.	Focus.	Tangent at vertex.
,,	Vertex.	Cissoid.
Central Conic.	Focus.	Auxiliary Circle.
,, ,,	Centre.	$r^2 = a^2\cos^2\theta \pm b^2\sin^2\theta$.
Rect. Hyp.	Centre.	Lemniscate.
Eq. Spiral (α).	Pole.	Eq. Spiral (α).
$(r/a)^m = \frac{\cos}{\sin} m\theta$.	Pole	$(r/a)^{\frac{m}{m+1}} = \frac{\cos}{\sin} m\theta/(m+1)$.

(2) **Inverse Curves**: *To find the Inverse.*

(*a*) *p, r-Form.* $PQQ'P'$ cyclic.

$\therefore$ ϕ of curve $= \pi - \phi$ of inverse.

$$\left.\begin{array}{l} \therefore\ p'/r' = p/r \\ rr' = a^2 \\ f(r, p) = 0 \end{array}\right\} \text{Elim. } r \text{ and } p.$$

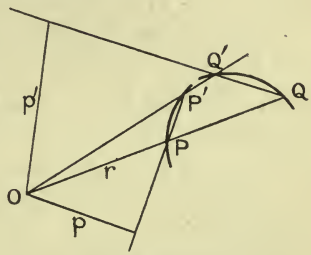


Fig. 49.

(*b*) *Polar Form.* If $f(r, \theta) = 0$ be the curve, $f(a^2/r, \theta) = 0$ is Inverse.

(*c*) *Cartesian Form.* If $f(x, y) = 0$ be the curve

$$f\left(\frac{a^2x}{x^2+y^2}, \frac{a^2y}{x^2+y^2}\right) = 0 \text{ is Inverse.}$$

(*d*) *Common Inverses.* (See also *GEOM.* § 33.)

Curve.	*Origin.*	*Inverse.*
Parabola.	Focus.	Cardioid.
,,	Vertex.	Cissoid.
Rect. Hyp.	Centre.	Lemniscate.
Central Conic.	Focus.	Limaçon.
,, ,,	Centre.	Cassinian Oval.

Note also that curves having nth contact at P invert into curves having nth contact at Q; that the osculating circle at P inverts into

a circle osculating the inverse curve at Q, but if the centre of inversion is on the circle, into a st. line tangent at Q, a point of inflexion on the inverse curve.

(3) **Polar Reciprocal.**

(a) Y' is taken so that $OP \cdot OY' = a^2$ and $Y'Q$ is drawn at rt. angles to OY'; then the Polar Reciprocal is the envelope of $Y'Q$.

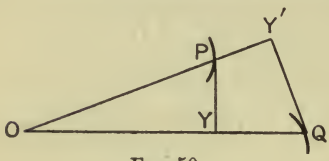


FIG. 50.

Or, which is the same thing, OY is drawn perpendicular to the tangent at P and Q taken such that $OY \cdot OQ = a^2$, P. R. is locus of Q.

(b) *p, r-Form.* The P. R. is clearly the inverse of Pedal. Use § 21 (1) (a), (2) (a).

(c) *Cartesian Form.* Use § 21 (1) (c), then (2) (b). Result is in Polars.

(d) *Common Polar Reciprocals.* (See also *G. CON.* § 35.)

Curve.	*Origin.*	*Inverse.*
Eq. Spiral (α).	Pole.	Eq. Spiral (α).

(4) **Caustics.** OP, PQ are drawn making equal angles with the tangent at P, the caustic is the envelope of PQ.

Common Caustics.

O at ∞, reflecting curve a circle, caustic an *epicycloid* (radii $2a$, a).
O on reflecting circle, caustic a *cardioid* touching circle at O.

22. ASYMPTOTES. i. *Cartesian Coordinates.*

(1) A curve of the nth degree has n asymptotes, real or imaginary. If n is odd one of these is necessarily real.

(2) *Asymptotes by inspection.* [u_n, v_n = homogeneous n-tics.]

(a) $(ax + by)\, u_{n-1} + v_{n-2} + v_{n-3} + \ldots = 0.$

Asymptote is $ax + by = 0$.

(b) $(ax + by)\, u_{n-1} + v_{n-1} + v_{n-2} + \ldots = 0.$

Asymptote is $ax + by = -\text{Lt.}\, \dfrac{v_{n-1}}{u_{n-1}}$.

(*c*) $(ax+by)^2 u_{n-1} + v_{n-1} + v_{n-2} + \ldots = 0.$

Asymptotes are $(ax+by)^2 = -\text{Lt.}\dfrac{v_{n-1}}{u_{n-1}}$.

(*d*) $(ax+by)^2 u_{n-1} + (ax+by) v_{n-1} + w_{n-1} + w_{n-2} + \ldots = 0.$

The asymptotes are

$$(ax+by)^2 + (ax+by)\,\text{Lt.}\frac{v_{n-1}}{u_{n-1}} + \text{Lt.}\frac{w_{n-1}}{u_{n-1}} = 0.$$

In all the above cases the limits ($x = \infty$, $y = \infty$) are to be taken subject to the limitation $\dfrac{x}{y} = -\dfrac{b}{a}$.

(*e*) If F_n be a function containing terms of nth degree *and lower*, and break up into linear factors, no two being parallel, all these lines are asymptotic to

$$F_n + F_{n-2} = 0.$$

(3) *Asymptotes parallel to Axes.* Equate to zero the coefficient of the highest powers of x, then that of the highest power of y. The resulting lines, if any, will be the asymptotes, and will give all asymptotes parallel to the axes.

(4) *Partial Fractions Method.* Arrange the equation in the form

$$x^n \phi\left(\frac{y}{x}\right) + x^{n-1}\psi\left(\frac{y}{x}\right) + \ldots = 0$$

[i.e. with terms of nth degree first, followed by terms of $(n-1)$th].

Then if $-\dfrac{\psi(t)}{\phi(t)}$ becomes, in partial fractions,

$$\frac{c_1}{t-m_1} + \frac{c_2}{t-m_2} + \ldots,$$

the asymptotes are $y = m_1 x + c_1$, $y = m_2 x + c_2$, &c.

(5) The rule of (4) is simply a convenient way of stating and developing the following:

Asymptotes, not parallel to axes, may be determined by substituting $y = mx + c$ in the equation to the curve, and equating to zero the coefficients of the two highest powers of x, thus giving two equations for m and c.

Note.—The limitation 'not parallel to axes' is important. Thus if we substitute $y = mx+c$ in $y^4 = y^2-x^2$ and apply the rule, the resulting equations are satisfied by $m = 0$ and $c = 0$, but $y = 0$ is not an asymptote.

(6) *Curvilinear Asymptotes.* Expansions of the forms

$$\begin{cases} y = mx + a + \dfrac{b}{x} + \dfrac{c}{x^2} + \dots, \\ y = mx + ax^{\frac{1}{2}} + b + cx^{-\frac{1}{2}} + \dots \end{cases}$$

will, by taking the first three terms, give hyperbolic or parabolic asymptotes respectively. Such expansions may be determined by substituting for y and equating successive powers of x to zero, but are usually best sought by successive approximations.

Notes.—(1) Of the two forms given, the second must be used when there are two parallel rectilinear asymptotes.

(2) *Example of Successive Approximations* (Edwards).

Curve $(y-x)^2(y+x) = 2ax^2$.

Hyperbolic Asymptote.

$$y+x = \frac{2ax^2}{(y-x)^2} = \frac{2ax^2}{4x^2} = \frac{a}{2}. \qquad (1st\ App.)$$

$$y+x = \frac{2ax^2}{(-x+\frac{a}{2}-x)^2} = \frac{a}{2} + \frac{a^2}{4x}. \qquad (2nd\ App.)$$

Parabolic Asymptote.

$$y-x = \sqrt{\frac{2ax^2}{y+x}} = \sqrt{\frac{2ax^2}{x+x}} = \sqrt{ax}. \qquad (1st\ App.)$$

$$y-x = \sqrt{\frac{2ax^2}{x+\sqrt{ax}+x}} = \dots = \sqrt{ax} - \frac{a}{4}. \qquad (2nd\ App.)$$

ii. *Polar Coordinates.* Find a value of θ, say α, which will make $r = \infty$ and $p = r^2 d\theta/dr$ finite. The asymptote is then the Cartesian line $y = x\tan\alpha - p_a \sec\alpha$.

iii. *Asymptotic Circle.* Examine if $\theta = \infty$ gives a definite limit for r, say a. Then $r = a$ is an asymptotic circle.

23. Singularities, &c.

(1) *Concavity or Convexity*:

(*a*) A curve is convex or concave to the foot of ordinate as $y\dfrac{d^2y}{dx^2}$ is + or −.

(*b*) A curve is convex or concave to the pole as $u + \frac{d^2u}{d\theta^2}$ is $-$ or $+$.

Note.—If the sign is forgotten it can be recovered in a moment by applying the test to the known curves $y = x^2$ or $u = 1/a$.

(2) Point of Inflexion (Flex).

Curve $y = \phi(x)$: $\phi''(x) = 0$ and changes sign at the point.

Polar Curve: $u + \frac{d^2u}{d\theta^2} = 0$ and changes sign at the point.

Note.—The appended table is a useful summary. It is sometimes, of course, more convenient to work with $\frac{d^2x}{dy^2}$ than with $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	
0			Tangent ∥ to x-axis.
0	$-$		Maximum.
0	$+$		Minimum.
0	0	$\neq 0$	Flex.
$\neq 0$	$+$		Convex downward.
$\neq 0$	$-$		Concave downward.

(3) Point of Undulation.

Curve $y = \phi(x)$: $\phi''(x) = 0$ and does *not* change sign at the point.

Polar Curve: $u + \frac{d^2u}{d\theta^2} = 0$ and does *not* change sign at the point.

(4) Double Points [Nodes, i.e. Crunodes, Acnodes, or Spinodes]. Any common solutions to

$$f(x, y) = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

give double points. These are:

Crunodes if $f_{xy}^2 > f_{xx} \cdot f_{yy}$ for the values in question.

Acnodes if $f_{xy}^2 < f_{xx} \cdot f_{yy}$.

Spinodes = Cusps if $f_{xy}^2 = f_{xx} \cdot f_{yy}$, usually. But sometimes acnodes.

If a spinode (cusp), its species (**keratoid** or **rhamphoid**) is usually best determined by transferring the origin to the point in question, but it can sometimes be settled by noting that for $\left\{\begin{matrix}\text{keratoid}\\ \text{rhamphoid}\end{matrix}\right\}$ cusps $\frac{d^2y}{dx^2}\left[\text{or } \frac{d^2x}{dy^2}\right]$ may have two values of $\left\{\begin{matrix}\text{opposite}\\ \text{same}\end{matrix}\right\}$ signs very near the cusp.

Notes.—(1) All the spinodes (cusps) lie on $f_{xy}^2 = f_{xx} \cdot f_{yy}$.

(2) The equation to the two tangents at a double-point (h, k) is

$$(x-h)^2 \frac{\partial^2 f}{\partial h^2} + 2(x-h)(y-k) \frac{\partial^2 f}{\partial x \partial y} + (y-k)^2 \frac{\partial^2 f}{\partial y^2} = 0.$$

(3) For a triple-point at (h, k) we must have

$$f = f_h = f_k = f_{hh} = f_{hk} = f_{kk} = 0,$$

and the equation to the three tangents is

$$(x-h)^3 \frac{\partial^3 f}{\partial h^3} + 3(x-h)^2(y-k) \frac{\partial^3 f}{\partial h^2 \partial k} + 3(x-h)(y-k)^2 \frac{\partial^3 f}{\partial h \partial k^2} + (y-k)^3 \frac{\partial^3 f}{\partial k^3} = 0.$$

(5) *The Fundamental Singularities.* All singularities on a rational curve may be resolved into nodes (∂), cusps (κ), flexes (ι), and bitangents (τ).

Ex. A Point of Undulation $= 2\iota + \tau$.

Note.—The rarer singularities which do not occur on rational integral algebraical curves are:

Point d'Arrêt, e.g. (0, 0) on $y = a^{\frac{1}{x}}$.

Branche Pointillée, e.g. in all quadrants except the first with the curve $y = x^x$.

(6) *Plücker's Equations.* If there be ∂ nodes, κ cusps, ι flexes, τ bitangents on a curve of degree n and class m [m = no. of tang. from ext. pt.],

$$\begin{aligned} m &= n(n-1) - 2\partial - 3\kappa, \\ n &= m(m-1) - 2\tau - 3\iota, \\ \iota &= 3n(n-2) - 6\partial - 8\kappa, \\ \kappa &= 3m(m-2) - 6\tau - 8\iota. \end{aligned}$$

24. Envelopes.

(1) If λ be a variable parameter,

Envelope of $A\lambda^2 + 2B\lambda + C = 0$ is $B^2 = AC$;

„ „ $A\cos\lambda + B\sin\lambda = C$ is $A^2 + B^2 = C^2$.

(2) The envelope of $\phi(x, y, \lambda) = 0$ is the λ-eliminant of

$$\phi(x, y, \lambda) = 0 \text{ and } \partial\phi/\partial\lambda = 0.$$

But this eliminant contains also as factors the locus of any nodes of $\phi(x, y, \lambda) = 0$, repeated twice, and the locus of any cusps, repeated thrice. If, therefore, the eliminant contains any square or cube factors, these should be carefully examined.

(3) The envelope of $\phi(x, y, \alpha, \beta) = 0$, where $f(\alpha, \beta) = 0$ is the eliminant of α, β, $\dfrac{d\beta}{d\alpha}$ from the following:—

$$\left\{\begin{array}{ll} \phi = 0, & f = 0, \\ \dfrac{\partial\phi}{\partial\alpha} + \dfrac{\partial\phi}{\partial\beta}\cdot\dfrac{\partial\beta}{\partial\alpha} = 0 & \\ \dfrac{\partial f}{\partial\alpha} + \dfrac{\partial f}{\partial\beta}\cdot\dfrac{\partial\beta}{\partial\alpha} = 0 & \end{array}\right\}.$$

The elimination is often best effected between

$$\phi = 0,\ f = 0,\ \frac{\partial\phi}{\partial\alpha} = \lambda\frac{\partial f}{\partial\alpha},\ \frac{\partial\phi}{\partial\beta} = \lambda\frac{\partial f}{\partial\beta}.$$

[*'Method of Undetermined Multipliers.'* Cf. § 15.]

(4) The envelope of $\phi(x, y, \alpha, \beta, \gamma) = 0$, where

$$f_1(\alpha, \beta, \gamma) = 0,\ f_2(\alpha, \beta, \gamma) = 0,$$

is the eliminant of α, β, γ, λ, μ between

$$\phi = 0,\ f_1 = 0,\ f_2 = 0,$$

$$\left\{\begin{array}{l} \dfrac{\partial\phi}{\partial\alpha} + \lambda\dfrac{\partial f_1}{\partial\alpha} + \mu\dfrac{\partial f_2}{\partial\alpha} = 0 \\ \dfrac{\partial\phi}{\partial\beta} + \lambda\dfrac{\partial f_1}{\partial\beta} + \mu\dfrac{\partial f_2}{\partial\beta} = 0 \\ \dfrac{\partial\phi}{\partial\gamma} + \lambda\dfrac{\partial f_1}{\partial\gamma} + \mu\dfrac{\partial f_2}{\partial\gamma} = 0 \end{array}\right\}.$$

25. CURVATURE.

(1) $\rho = ds/d\psi$.

(2) $\rho = (1+p^2)^{\frac{3}{2}}/q$, where $p = \dfrac{dy}{dx}$, $q = \dfrac{d^2y}{dx^2}$.

(3) $\rho = rdr/dp$, where $p = \perp^{\mathrm{r}}$ on tangent from pole.

(4) $\rho = p + \dfrac{d^2p}{d\psi^2}$.

(5) ρ at origin = Lt. $x^2/2y$ if $y = 0$ be the tangent.
= Lt. $y^2/2x$ if $x = 0$,, ,,

Other formulae of less importance are added for reference :

(6) $\dfrac{1}{\rho} = -\dfrac{\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}} = \dfrac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}}$.

(7) $\dfrac{1}{\rho^2} = \left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2$.

(8) $\rho = \dfrac{(p^2+q^2)^{\frac{3}{2}}}{rq^2-2spq+tp^2}$. Curve $f(x, y) = 0$ $\begin{cases} p = f_x,\ q = f_y \\ r = f_{xx},\ s = f_{xy},\ t = f_{yy} \end{cases}$.

(9) $\rho = \dfrac{(F_1^2+f_1^2)^{\frac{3}{2}}}{F_1f_2-F_2f_1}$. Curve $\begin{cases} x = F(t) \\ y = f(t) \end{cases}$.

(10) $\rho = \dfrac{(u^2+u_1^2)^{\frac{3}{2}}}{u^3(u+u_2)}$. $u = \dfrac{1}{r}$, $u_1 = \dfrac{du}{d\theta}$, $u_2 = \dfrac{d^2u}{d\theta^2}$.

(11) $\rho = \dfrac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$. $r_1 = \dfrac{dr}{d\theta}$, $r_2 = \dfrac{d^2r}{d\theta^2}$.

(12) Centre of Curvature : $\begin{cases} \bar{x} = x - \rho\sin\psi, \\ \bar{y} = y + \rho\cos\psi. \end{cases}$

(13) Two curves have contact of the second order at a point of intersection where $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ have the same value for both curves ; contact of the third order when, in addition, $\dfrac{d^3y}{dx^3}$ has the same value.

(14) **Conic of Closest Contact.** The conic through *3 consecutive points* on the curve will have ρ the same at the points as on the curve; through *4 consecutive pts.* will have ρ and $\frac{d\rho}{ds}$ the same; through *5 consecutive pts.* ρ, $\frac{d\rho}{ds}$, $\frac{d^2\rho}{ds^2}$ the same. Now

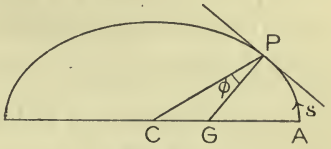


Fig. 51.

in any conic $\tan\phi = \frac{1}{3}\frac{d\rho}{ds}$, where ϕ is the angle *in front* of the normal as regards the direction in which s is measured. This gives the direction of the centre of all conics of 4-point contact $\left(\rho \text{ and } \frac{d\rho}{ds} \text{ being taken from the curve}\right)$ or of the axis of *the* parabola of 4-pointic contact.

For the centre of the **Osculating Conic** (of 5-point contact) we measure along PC (which is drawn at the inclination ϕ in front of PG) a distance r given by

$$\frac{\cos\phi}{r} = \frac{1}{\rho} - \frac{d\phi}{ds}.$$

Here again ϕ, ρ, $\frac{d\phi}{ds}$ are found *from the curve.*

(15) **Evolutes.** To find evolute of a curve find the envelope of the normal.

26. Curve Tracing, Cartesian Coordinates.

i. Note any *Symmetry.*

With regard to Ox: y occurs in *even* powers only.

" " " Oy: x " " " "

" " " $x = y$: Unaltered by interchange of x and y.

" " " $x = -y$: Unaltered by interchange of x and $-y$.

In opposite quadrants. Unaltered by writing $-x$, $-y$ for x, y.

If not symmetrical it may sometimes be rendered so by some obvious transformation of coordinates.

Ex. $y^3 = x^2 - 4x + 4$, $(2x - y)^2 = (x + 2y - 3)^3$.

ii. *Differentiate* and find $p = \dfrac{dy}{dx}$.

Thus if $\phi(x, y) = 0$, $p = -\phi_x/\phi_y$.

If $\phi_x = 0$, $\phi_y = 0$ are simple curves it will be a useful confirmatory test to trace them. Wherever $\phi_x = 0$ intersects $\phi = 0$, $p = 0$, and wherever $\phi_y = 0$ intersects $\phi = 0$, $p = \infty$. Wherever $\phi_x = 0$ intersects $\phi_y = 0$ we have a Singular Point, and the intersections of these two give all the singular points except flexes.

iii. *Tabulate* x, y, and p for any points which obviously lie on the curve and any singular points which may have been found, finding p for each. In particular find all the points where the curve meets $x = 0$, $y = 0$, and $x = \pm y$.

There are two cases in which any number of points may be determined and the curve traced by this method alone with tolerable accuracy:

(1) When x can be found in terms of y, or y in terms of x.

(2) When, after substituting $y = mx$, x can be found in terms of m. This method can always be applied when the dimensions of the terms do not differ by more than 2. This form always allows us to determine the existence of infinite branches.

Ex. $x^3 + y^3 = 3axy$, $x = \dfrac{3am}{1+m^3}$, $y = \dfrac{3am^2}{1+m^3}$; $\therefore p = \dfrac{2m - m^4}{1 - 2m^3}$, $m = -1$ gives points at ∞. This is the *Folium of Descartes*.

iv. When it is possible to *solve for* x (in terms of y) or to *solve for* y:

(1) The solution will usually show limits between which the curve either must, or cannot lie, and thus usually detect loops.

Ex. $x^4 = x^2 - y^2$, $\therefore x^2 > y^2$, $\therefore$ curve lies between $y = +x$ and $y = -x$ in the quadrant in which $y = 0$ lies;

also $y^2 = x^2 - x^4 = x^2(1-x^2)$, $\therefore x < 1$;

also $2x^2 = 1 \pm \sqrt{1-4y^2}$, $\therefore y < \pm \frac{1}{2}$.

(2) We can often then detect conjugate points as points lying between the forbidden limits.

Ex. $y^2 = x^2 \dfrac{x+1}{x-1}$, $\therefore$ curve does not lie between $x = 1$ and $x = -1$. But (0, 0) is on the curve, $\therefore$ it is an *acnode*.

(3) The solution by expansion will immediately give rectilinear and curvilinear asymptotes.

Ex. $y^2 = x^2 \cdot \dfrac{x+1}{x-1} = x^2\left(1+\dfrac{1}{x}\right)\left(1-\dfrac{1}{x}\right)^{-1} = x^2\left(1+\dfrac{2}{x}+\ldots\right)$,

$$\therefore y = \pm x\left(1+\frac{1}{x}\right) = \pm(x+1).$$

(4) The solution will frequently give a diametral line or curve.

Ex. $x^2 - y^2 = 2x(xy+1)$, $\therefore y = -x^2 \pm \sqrt{x^4+x^2-2x}$, $\therefore y = -x^2$ bisects all chords parallel to $x = 0$.

v. Find all the *asymptotes* [see § 22] and investigate whether the curve lies above or below them.

Note.—*Always* investigate also whether the curve cuts its asymptotes. If it does, enter the intersections in the table in iii.

In the case of a cubic curve the intersections with the three asymptotes lie on a st. line, whose equation is easily found and forms a valuable verification of the correctness of the drawing.

Ex. $xy(x+y) = x^2+y^2$. Asymptotes are $x = 1$, $y = 1$, $x+y = -2$; hence, the line is $xy(x+y) - x^2 - y^2 - (x-1)(y-1)(x+y+2) = 0$, i. e. $x+y-2 = 0$.

vi. *Singular Points.* The possible existence of singularities may have been detected in ii by the fact that $p = -\dfrac{\phi_x}{\phi_y}$ assumes the form $\dfrac{0}{0}$, and therefore [see § 11 (2) (*a*)]

$$= -\frac{\phi_{xx} + \phi_{xy} \cdot p}{\phi_{yx} + \phi_{yy} \cdot p}.$$

If this is not still indeterminate it will give a quadratic for the value of p at the singularity. When the position of a singularity has been determined, its nature is best examined by transferring the origin to the singularity and using vii.

Caution.—In this quadratic for p be careful not to miss the value $p = \infty$ which occurs when the quadratic reduces to a simple equation.

vii. *Newton's Rule.* [Form of Curve at Origin and Infinity.] Let each term of the equation be represented on squared paper by a point whose coordinates are its x and y indices.

Thus $5x^2y^3$ is entered as the point (2, 3). If a straight line can be drawn joining two or more of these points, and *not passing through O*, which *shuts off all other points* from *O*, the terms of the equation represented by this line give the form of the curve at the origin. If a st. line can be drawn shutting off *no point* from the origin the terms represented will give the form at ∞, i.e. will give curves more or less asymptotic (rarely an exact asymptote).

Notes.—(1) *Conjugate Points* (Acnodes) at the origin are indicated by the 'Form at the Origin' becoming wholly imaginary.

(2) *Absence of Infinite Branches* is indicated by the 'Form at Infinity' becoming wholly imaginary.

(3) *Singular Points* may be studied by transferring the origin to them and using Newton's Rule.

(4) *Ex.* Newton's Rule applied to curve $y^4 = x + xy + x^3y^2$ shows that $y^4 = x$ is the form at the origin and $y^2 = x^3$, $x = x^3y^2$, i.e. $xy = \pm 1$ is the 'form at ∞'. The rule applied to $y^2(x-1) = x^2(x+1)$ gives for form at *O*, $x^2 + y^2 = 0$ and indicates an acnode.

viii. The *Tentative Tracing* for which sufficient details probably now exist can be examined by the following considerations:

(1) By turning into polars and solving for r in terms of θ. This will usually detect loops.

(2) Any straight line intersects a curve of the nth degree in n points, real or imaginary, and imaginary roots occur in pairs.

(3) There cannot be *single* branches going to infinity.

(4) In particular, if a branch approaches an asymptote, there must be a second branch approaching the same asymptote either at the same or at the opposite end.

ix. *Special Devices.*

(1) *Auxiliary Loci.*

Ex. (1) $y(y^2-ax) = (y^2+a^2)(x-a)$. It is easy to see that the curve lies wholly in the regions of space marked +, and passes through the points marked •. When the asymptote $(x = y)$ is drawn the curve is now easily traced.

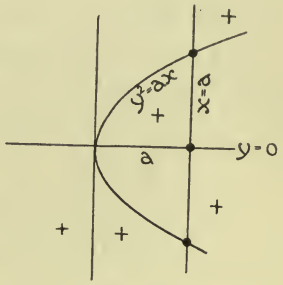


FIG. 52.

Ex. (2) $(y-x^2)^2 = x^5$, $\therefore y = x^2 \pm x^{\frac{5}{2}}$. Trace $y = x^2$, $y = x^{\frac{5}{2}}$ separately, and add and subtract their ordinates.

(2) *Transformation of Coordinates.* [Cf. *Note* on § 26, i.]

Ex. $(x+y+1)^2 = (y-x+3)^5$. Refer to new axes $\begin{cases} x+y+1 = 0. \\ y-x+3 = 0. \end{cases}$

So also with $(x+2y+1)^2 = (3y+x-5)^5$, even although the new axes in this case are oblique.

27. CURVE TRACING, POLAR COORDINATES.

i. Note any *Symmetry.*

If change of sign of θ leaves equation unaltered: Initial line.

If $\theta + \pi$ for θ leaves equation unaltered: Opposite quad^s.

If $\pi - \theta$ for θ „ „ „ : Axis of y.

If not symmetrical it may sometimes be rendered so by some obvious transformation.

Ex. $r = a(1+\sin 2\theta)$ has at present symmetry in opposite quad^s.

$= a(\sin\theta+\cos\theta)^2 = 2a\sin^2\left(\theta+\frac{\pi}{4}\right)$, showing symmetry with regard to the line $\theta + \frac{\pi}{4} = 0$.

ii. Note any limitations imposed on r by the trigonometrical functions of θ.

Ex. $r = a \sin n\theta$, $\therefore r \not> a$, $\therefore$ curve wholly inside circle $r = a$.

iii. Find the *asymptotes* [§ 22, ii] by finding the value of θ which makes $r = \infty$ but $\dfrac{d\theta}{du}$ finite.

If r can be determined explicitly in terms of θ it is now easy to trace the curve by making θ gradually *increase* from 0 up to the value (if any) when r repeats itself, and then *decrease* from 0.

iv. In any case the values of r must be found for selected values of θ $\left(\text{particularly } 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\right)$, and ϕ calculated from $\tan\phi = \dfrac{rd\theta}{dr}$. [Remember that ϕ and θ are *on the same side of* r, and then selected points must be tabulated, θ, r, and $\tan\phi$ being entered in the table.]

v. Note (1) $r = \infty$. The curve has infinite branches.
(2) r finite, $\theta\,\infty$. Asympt. circle.
(3) *Points of Inflexion.* [See § 23 (2).]
(4) *Concavity or Convexity.* [See § 23 (1).]

28. The well-known Higher Curves.

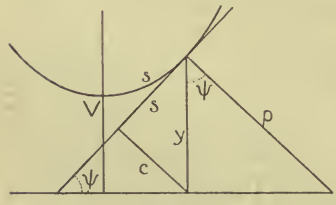


Fig. 53.

(1) Chainette or Catenary. All the properties may be deduced from the figure, s being measured from r.

Cartesian Equation:

$$y = c \cosh (x/c).$$

Note:

{ Catenary: $\rho = -$ normal. }
{ Circle: $\rho = +$ normal. }

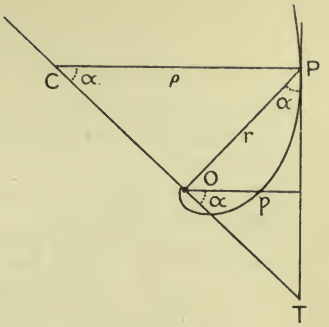


Fig. 54.

(2) **Equiangular or Logarithmic Spiral.**

$PC = \rho$, $PT = s$.

Equation :

$$r = ae^{\theta \cot \alpha}.$$

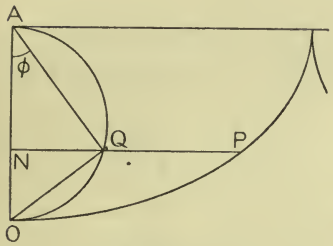


Fig. 55.

(3) **Cycloid.**

Tangent at P is $\parallel$ to OQ.

Arc OP = 2 chord OQ.

Arc OQ = PQ.

ρ_P = 2 chord AQ.

Area = 3 × area of generating circle.

The Cartesian Equation is easily written in terms of $\theta = 2\phi$. See (6) below.

(4) **Epicycloids** } have all equations of the type
Hypocycloids }

$$p^2 = A + Br^2.$$

They are the θ-eliminants of equations of the type

$$\begin{cases} x = p\cos\theta - q\cos r\theta, \\ y = p\sin\theta - q\sin r\theta. \end{cases}$$ [See (6) below.]

(5) Of the other curves the following details should be remembered :

Trochoids are θ-eliminants of $\begin{cases} x = a\theta + b\sin\theta, \\ y = a\theta - b\cos\theta. \end{cases}$

When $a = b$ we have a cycloid.

Cartesian Ovals: $lr_1 + mr_2 = a.$

Cassinian Ovals: $r_1 r_2 = a^2.$

Folium of Descartes: $x^3 + y^3 = 3axy.$ [See §26, iii, *Ex.*]

Lemniscate: $r^2 = a^2 \cos 2\theta.$ [See (6) below.]

Cardioid: $r = a(1 + \cos\theta).$ [See (6) below.]

Limaçon: $r = a\cos\theta + b.$ [See (6) below.]

Circular Cubics: $u_2 + u_1(x^2 + y^2) = 0.$

Bicircular Quartics: $u_2 + u_1(x^2 + y^2) + (x^2 + y^2)^2 = 0.$

Semicubical Parabolas: $y^2 = k(x-a)(x-b)(x-c).$

(6) The following are added for reference:

Lemniscate: Is both central inverse and the central pedal of a rectangular hyperbola.

Cardioid / *Limaçon* } Pedal of circle, pedal origin { on / not on } circumference.

also, inverse of { parabola / conic, eccentricity a/b } with regard to focus.

also, { epicycloid / epitrochoid } when radii of fixed and moving circles are equal. The limaçon $r = 1 + 2\cos\theta$ is the *Trisectrix*.

Tractory / *Tractrix* } Is a curve such that the intercept of the tangent between the point of contact and a fixed line is constant. It is the involute of the catenary, and the locus of T in the catenary figure, and its properties may be derived from the catenary figure. Equation $x = c\log[c + \sqrt{c^2 - y^2}] - c\log y - \sqrt{c^2 - y^2}.$

Syntractrix: Locus of a fixed point on the tangent to the tractrix. Equation $x = c\log[d + \sqrt{d^2 - y^2}] - c\log y - \sqrt{d^2 - y^2}.$

Cycloid: $x = a\sin\theta + a\theta,$
$y = a(1 - \cos\theta).$

Epicycloids, &c: If a, b are the radii of the fixed and moving circles the values for the epi-curves of p and r [see (4) above] are $p = a + b$, $r = (a+b)/b$, while for the hypo-curves the sign of b is changed. For the epi- or hypocycloid $p = qr$, otherwise we have epi- or hypotrochoids. The epi- and hypocycloids are all included in $p^2 = k(r^2 - a^2)$, where $k = \frac{9}{8}, \frac{4}{3}, \frac{25}{16}, \frac{9}{5}$ for the epicycloids with 1, 2, 3, 4 cusps respectively, and $k = \infty, 0, -\frac{1}{8}, -\frac{1}{3}$ for the corresponding hypocycloids, the first of these being the fixed circle itself, and the second one of its diameters. The *Four-Cusped Hypocycloid* is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

Companion to Cycloid: $x = a\theta$
$y = a(1 - \cos\theta)$ } is a harmonic curve.

Harmonic Curve / *Curve of Sines*: $y = a \sin mx$.

Logarithmic Curve: $y = a \log x$, $y = ae^{kx}$.

Probability Curve: $y = ae^{-kx^2}$.

Cissoid of Diocles: $y^2(2a-x) = x^3$ is the first positive pedal with regard to the vertex, and also the inverse with regard to vertex of a parabola.

Witch of Agnesi: $xy^2 = 4a^2(2a-x)$.

Quadratrix of Dinostratus: $y = x \cot \frac{\pi x}{2a}$.

,, ,, *Tschirnhausen*: $y = a \cos \frac{\pi x}{2a}$.

Conchoid of Nicomedes: $r = a \operatorname{cosec} \theta \pm b$.

Involute of Circle: $r^2 = p^2 + a^2$.

The Spirals:

Spiral of Archimedes:	$r = k\theta$.	These belong to the class of Parabolic Spirals $r = k\theta^n$.
Hyperbolic / *Reciprocal* *Spiral*:	$r = k\theta^{-1}$.	
Lituus:	$r = k\theta^{-\frac{1}{2}}$.	

The p-r equation of the above is easily found.

Cotes's Spirals arise in Dynamics. Their equation is

$$\frac{1}{p^2} = \frac{a}{r^2} + b.$$

This class includes Equiangular Spiral, Reciprocal Spiral, and the Curves $u = a \frac{\sinh}{\cosh} n\theta$, $u = a \frac{\sin}{\cos} n\theta$.

ELEMENTARY STATICS

1. (1) *Parallelogram of Forces.* If two st. lines be drawn from a point representing completely (i.e. in magnitude, direction, and sense) forces acting on a particle, and a parallelogram be constructed having these two lines as adjacent sides, the diagonal drawn from the point named will represent the resultant completely.

(2) *Triangle of Forces.* If three forces acting on a particle can be represented completely by the sides of a triangle taken in order the forces are in equilibrium, AND CONVERSELY.

Hence forces represented by AB, BC are equivalent to force represented by AC.

(3) *Polygon of Forces.* If any number of forces acting on a particle can be represented completely by the sides of a polygon taken in order the forces are in equilibrium, AND CONVERSELY.

Note.—If a system of forces in equilibrium be parallel to the sides of a polygon taken in order they are *not* necessarily proportional to them. Further, the polygon of the theorem need not be plane.

(4) *Parallelepiped of Forces.* Cf. § 1 (1).

(5) *Components*: $X = P\cos\theta.$
$Y = P\sin\theta.$

Note.—P makes angle θ with X, $90°-\theta$ with Y.

2. RESULTANTS. (1) $R^2 = P^2 + Q^2 + 2PQ\cos\alpha.$

Note.—The direction of R is inclined to that of P at an angle θ given by $\tan\theta = Q\sin\alpha/(P+Q\cos\alpha)$. Obtained by resolving Q into $Q\cos\alpha$ along P, $Q\sin\alpha$ $\perp^r$.

(2) $R^2 = (\Sigma X)^2 + (\Sigma Y)^2.$

(3) $R^2 = \Sigma P^2 + 2\Sigma PQ \cos \angle PQ$, for any number of forces, whether coplanar or not. [Cf. *SPH. TRIG.* § 10.]

(4) Resultant of OA, OB is $2OG$, where G lies in AB so that $AG = BG$.

(5) Resultant of $m \,.\, OA$, $n \,.\, OB$ is $(m+n)\, OG$, where G lies in AB so that $m \,.\, AG = n \,.\, BG$.

(6) Resultant of OA_1, OA_2 ... OA_r is $r \,.\, OG$, where G is the centroid of $A_1 A_2 \ldots A_r$.

(7) Resultant of $m_1 \,.\, OA_1$, $m_2 \,.\, OA_2$ ... $m_r \,.\, OA_r$ is $(\Sigma m)\, OG$, where G is the centroid of m_1 at A_1, m_2 at A_2, ... m_r at A_r.

(8) *Lami's Theorem.* If P, Q, R equilibrate at a point,

$$\frac{P}{\sin \angle QR} = \frac{Q}{\sin \angle RP} = \frac{R}{\sin \angle PQ}.$$

3. Theorem of Moments. The algebraical sum of the moments of any number of $\left\{\begin{matrix}\text{coplanar forces}\\ \text{forces}\end{matrix}\right.$ about a $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ is equal to the moment of their resultant about the same, and is zero if the forces are in equilibrium.

4. Conditions of Equilibrium. (Concurrent coplanar forces.) A system of concurrent coplanar forces is in equilibrium if

(1) The sum of the resolutes in each of two directions is zero.

or (2) The sum of the moments about each of two points is zero provided that the join of these points does not pass through the point at which the forces act.

5. Elastic Strings.

(1) *Hooke's Law.* Tension of an elastic string

$$= \lambda\left(\frac{\text{Extension}}{\text{Length}}\right).$$

λ is the 'modulus' for that particular string = tension required to double its length.

Note.—The value of λ is independent of the length of the string and varies directly with the cross-section. λ for a string (wire) 1 cm.^{2} in section is known as 'Young's Modulus' for the material of the string.

(2) *Work* done in *stretching* an elastic string
= (Mean of Tensions) × (Extension).

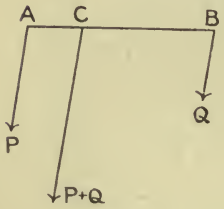


Fig. 56.

6. Parallel Forces.

(1) *Like.* The resultant of P and Q is $P+Q$, acting so that if ACB be any transversal cutting the forces in A and B and the resultant in C, $P \,.\, AC = Q \,.\, BC$.

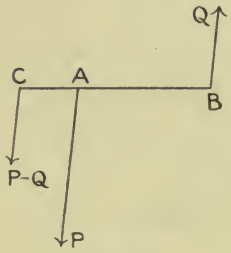


Fig. 57.

(2) *Unlike.* The resultant of P and Q is $P-Q$, acting nearer to the greater force and so that

$$P \,.\, AC = Q \,.\, BC.$$

7. Couples.

(1) Equimomental couples in the same or in parallel planes are equivalent.

(2) The resultant of any number of couples in parallel planes is a couple whose moment is equal to the sum of the moments of the couples.

(3) Axes of couples follow the Parallelogram Law.

(4) Any system of forces acting on a rigid body can be reduced to a single force acting at any arbitrary point and a couple.

(5) Any system of forces acting on a rigid body may be replaced by a single force and a couple whose axis is along the direction of that force (Poinsot's Central Axis). This central axis is the line about which the sum of the moments of all the forces is a minimum.

8. Conditions of Equilibrium of a Rigid Body.

(1) If three forces maintain equilibrium they must be coplanar, and either all concurrent or all parallel.

Note.—In the application of this result the following trigonometrical results are of great value :

(1) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta.$

(2) $\qquad\qquad = n\cot A - m\cot B.$

The general form only of these equations should be remembered, and the equations found more exactly when required.

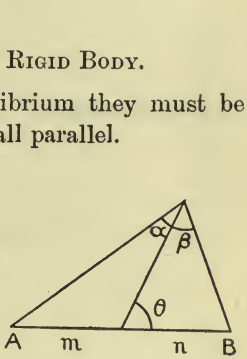


Fig. 58.

(2) Any number of coplanar forces will be in equilibrium if, and only if :

(*a*) { Moment about any one point is zero.
Components in any two directions are zero.

or (*b*) { Moments about any two points are zero.
Components in any one direction (not at rt. angles to the join of the two points) are zero.

or (*c*) Moments about any *3* non-collinear points are zero.

or (*d*) If the forces are $(X_1, Y_1)(X_2, Y_2)\ldots$ acting at (x_1, y_1) $(x_2, y_2)\ldots$, then $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma(xY - yX) = 0$.

(5) Any number of forces will be in equilibrium if, and only if :

(*a*) { Resolutes in any three non-coplanar directions are zero.
Moments about these or about any three non-coplanar axes are zero.

(*b*) When the forces are $(X_1, Y_1, Z_1)\ (X_2, Y_2, Z_2)\ \ldots$ acting at $(x_1, y_1, z_1)\ (x_2, y_2, z_2)\ \ldots$,

$$\begin{cases} \Sigma X = \Sigma Y = \Sigma Z = 0, \\ \Sigma(yZ - zY) = \Sigma(zX - xZ) = \Sigma(xY - yX) = 0. \end{cases}$$

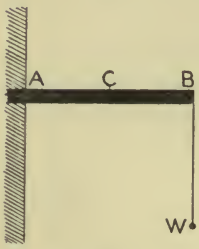


Fig. 59.

9. Actions at Connecting Points.

i. *Rigid Connexion.* If the connexion be rigid (as at any point of a rod) the action across this connexion is equal and opposite to the resultant of *all the forces on one side* (either side) *of the connexion* and consists of two parts :

(*a*) A force acting through the point.

'Tension' / 'Thrust' } = component of this force along rod.

'Shear' = component of this force $\perp^r$ to rod.

(*b*) A couple whose moment is equal and opposite to the moment of all the forces on either side of the point. This couple is known as the 'tendency to bend', 'bending stress', 'shear-couple', 'moment of flexure'. The rod will break, if at all, at the point where this couple is a maximum, provided it is a rod of uniform thickness.

Note.—Thus in Fig. 59 we have for the action at C :

Tension → $= 0$.

Shear ↑ $= W + w \cdot BC$ (w = wt. of unit length).

Shear-couple ↻ $= W \cdot BC + \frac{1}{2} w \cdot BC^2$.

ii. *Smooth Joint.* In this case the shear-couple is zero and the force is generally resolved in two convenient directions at rt. angles. Such resolutions are shown in the figures. In the second figure we think of both rods as jointed to a pivot to which the string is tied. If the string in this case be replaced by a rigid bar the actions at the hinge will usually be indeterminate.

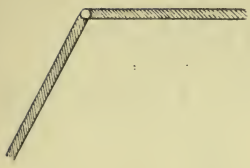


Fig. 60.

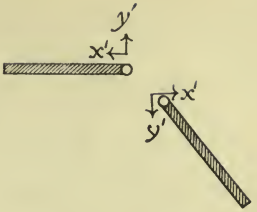


Fig. 61.

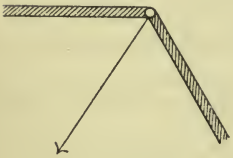


Fig. 62.

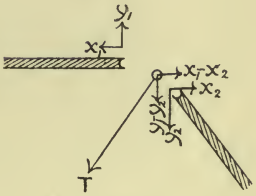


Fig. 63.

Notes.—(1) *Indeterminate Reactions.* It may be said, generally, that the reaction along a rod or bar of a connected system (framework, &c.) is determinate whenever the length of that rod can be increased *without undoing the frame.*

(2) The 'tendency to break' at any point of a rod is, as usual, the moment of all the forces on one side of the point, e. g. in Fig. 61 for the upper rod it is the moment of y_1 only, if the rod is light.

(3) *Virtual Work* (§ 15) is very useful for these problems.

10. Centre of Gravity.

i. **Lines**: (1) *Uniform St. Line.* Middle point.

(2) *Uniform Triangular Wire.* Incentre.

(3) *Arc of Circle.* Distance from centre

$$= \left[\frac{\sin \frac{1}{2}\text{angle}}{\frac{1}{2}\text{angle}}\right] \times \text{radius}.$$

Deduce *Semicircular Wire*, distance $= 2a/\pi$.

ii. **Areas**: (1) *Triangle.* Centroid, i. e. intersection

medians, one-third up median. Coincides with C. Gr. of three equal particles at angles.

(2) *Parallelogram.* Intersection of diagonals.

(3) *Sector.* Distance from centre $= \frac{2}{3}$ distance for arc.

Deduce *Semicircle*, distance $= \frac{4a}{3\pi}$.

Deduce (by Projection) *Semi-ellipse* (any diameter), distance $= \frac{4a}{3\pi}$, on conjugate.

(4) *Cone, Curved Surface.* One-third up axis from base.

(5) $\left\{\begin{matrix} Zone \\ Surface \end{matrix}\right\}$ *of Sphere*, cut off by parallel planes. Centre of median line. Deduce *Hollow Hemisphere.*

iii. **Volumes**: (1) *Pyramid on any Base.* One-quarter up join of centroid of base to vertex.

(2) *Cone on any Base.* One-quarter up join of centroid of base to vertex.

(3) *Hemisphere.* Distance from centre $= \frac{3}{8}r$.

(4) *Sector of Sphere.* [Reduce to ii (5).]

Deduce *Segment of Sphere.* [Use iii (2).]

Also *Zone* (Vol.) *of Sphere.* [Diff. of 2 segments.]

(5) *Paraboloid of Revolution.*

Distance $= \frac{2}{3}h$ from vertex.

Note.—For a tetrahedron the volume-centroid corresponds with that of four equal particles at the angles.

11. Determination of C. Gr.: General Theorems.

(1) $x = \Sigma m_1 x_1 / \Sigma m_1$, $y = \Sigma m_1 y_1 / \Sigma m_1$, $[z = \Sigma m_1 z_1 / \Sigma m_1]$.

Notes.—(1) $r \neq \Sigma m_1 r_1 / \Sigma m_1$, for the r's are not parallel.

(2) For the solution of problems involving varying density the result $1^r + 2^r + \ldots + n^r = \frac{n^{r+1}}{r+1}$ + lower powers of n is needed.

(2) Given C. Gr. of the whole of a mass m (at G) and of a part m_1 (at O_1), to determine that of the remainder: Introduce the conception of negative mass and find the C. Gr. of $+m$ at G, and $-m_1$ at O_1.

12. Properties of C. Gr.: General Theorems.

(1) If masses $m_1\ m_2 \ldots m_n$ be placed at points $A_1 A_2 \ldots A_n$ (not necessarily coplanar) and G be their centroid, O any other point, $\Sigma m \,.\, AO^2 = \Sigma m \,.\, AG^2 + OG^2 .\, \Sigma m$.

Note.—This result is exceedingly useful geometrically.

(2) *Guldinus' Theorem.* See *MENS.* § 31.

Note.—Hence mental calculation gives C. Gr. of area and circumference of a semicircle, &c.

13. Laws of Friction (Experimental).

i. *General*: (1) The **direction** of friction between two bodies is opposite to that in which their point of contact is urged to move.

(2) The **magnitude** of friction is (when there is equilibrium) just sufficient to prevent motion.

(3) No more than a certain amount ('**Limiting Friction**') can be called into play.

ii. *Limiting Friction*: (1) The **magnitude** of L.F. bears for the same bodies a constant ratio (μ = '**Coefficient of Friction**') to the normal pressure between the bodies, i.e. $F = \mu R$.

(2) μ **is independent** of the area and shape of the bodies in contact.

Note.—The actual value of μ for metal-metal is about 0·2; for wood-wood, 0·3 to 0·5. It varies greatly according to circumstances.

iii. *Dynamical Friction*: (1) The friction called into play to prevent motion is greater than that called out to resist its continuance.

(2) For the same substances the friction is approximately independent of the rate of their relative motion.

iv. *Rolling Friction*: (1) When a body is on the point of rolling upon another body the action at the point of contact consists of
- (α) Normal Reaction.
- (β) Tangential Friction Force.
- (γ) Friction Couple.

(2) The moment of the friction couple is independent of the curvature and proportional to the normal pressure.

Note.—It is commonly stated that 'Rolling Friction' is less than 'Sliding Friction', which means, presumably, that the force (β) of iv (1) is less than the force F of ii. This statement is only true with wide limitations. Recent tramway accidents have shown that brakes which nearly but not completely lock the wheels are more effective than those that do so completely.

14. Friction (*Miscellaneous*).

(1) Problems in Friction are usually best treated by the use of the total reaction (which in the case of Limiting Friction makes an angle $\lambda = \tan^{-1}\mu$ with the normal). They can frequently be thus reduced to three-force problems.

Note.—λ is the angle of the inclined plane on which the body just rests under the action of its weight and of the reactions of the plane only.

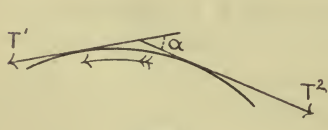


Fig. 64.

(2) For a string passing round a rough surface and in limiting equilibrium

$$T_1 = T_2 \cdot e^{\mu\alpha}.$$

(3) When equilibrium can be broken either by rolling or by sliding, rolling will occur unless it should involve a reaction at the point of rolling lying outside the cone of friction.

15. Virtual Work.

[Sometimes called '*Virtual Moment*'. The old name, 'Virtual Velocity', should never be used.]

If a force P act at a point A of a body along AN, and the point A move into a position A' very near to A,

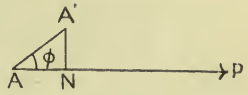


Fig. 65.

$P.\,AN = P.\,AA' \cos\phi$ = virtual work done by P in the displacement.

i. (1) If a number of forces act at a point A, the sum of the V. W. owing to the displacement of A = V. W. of the resultant.

(2) If a particle be in equilibrium under any number of forces acting at a point A, the sum of the V. W. for any displacement of A is zero.

(3) If a system be in equilibrium under the action of any number of forces and receive a small displacement consistent with the geometrical connexions between the bodies of the system, the sum of the V. W. for such a displacement is zero.

Note.—More strictly not zero but of the second order of small quantities.

ii. (1) The V. W. of the tension of an elastic string, whether straight or deflected (by smooth rings or pegs), is $-T.\,\delta l$.

(2) Provided that the geometrical conditions be not violated, we may consider the V. W. of the following to be zero:

Tensions of Inelastic Strings.

Mutual Attractions.

Interactions at Smooth Joints.

The Reaction (but NOT the Friction) when a body is virtually displaced by sliding along a rough surface.

The Reaction when a body rolls without sliding on

(a) a fixed surface, *or*

(b) a movable surface belonging to the same system.

16. Stability of Equilibrium.

(1) If a body be placed at rest but not in equilibrium under the action of external forces it will begin so to move that the work of the forces in the initial displacement is positive, i.e. so that the potential energy decreases.

(2) The equilibrium of a body, or system, is $\left\{\begin{matrix}\text{stable}\\\text{unstable}\end{matrix}\right\}$ according as the potential energy is a true $\left\{\begin{matrix}\text{minimum}\\\text{maximum}\end{matrix}\right\}$.

(3) The equilibrium of a system of bodies under the influence of their own weights and such constraints and connexions as are named in 15, ii is $\left\{\begin{matrix}\text{stable}\\\text{unstable}\end{matrix}\right\}$ when the height of the C. Gr. of the system above any fixed horizontal plane is $\left\{\begin{matrix}\text{minimum}\\\text{maximum}\end{matrix}\right\}$.

(4) Stable and unstable positions occur alternately for any system which moves with but one degree of freedom.

(5) If a body whose radius is R rests on another body of radius r, its C. Gr. being at height h vertically above the point of contact, the equilibrium is stable or unstable as

$$1/h \gtrless 1/R + 1/r.$$

ELEMENTARY DYNAMICS

1. (1) *Parallelogram of Velocities.* If a moving point possesses simultaneously velocities which are represented in magnitude, direction, (and sense), by the two sides of a parallelogram drawn from a point, its actual motion is with a velocity which is represented by the diagonal of the parallelogram through the point.

(2) *Triangle of Velocities.* If a point possesses simultaneously velocities represented in magnitude and direction by the sides of a triangle taken in order, the point is at rest.

Hence velocities represented by AB, BC are equivalent to velocity represented by AC.

(3) *Polygon of Velocities.* If a point possesses simultaneously velocities represented in magnitude and direction by the sides of a closed polygon taken in order, the point is at rest.

(4) *Parallelepiped of Velocities.* Cf. § 1 (1).

(5) *Components.* Any velocity u may be resolved into $u\cos\theta$ along a direction making an $\angle\,\theta$ with u and $u\sin\theta$ at rt. angles to that direction.

2. Resultants. (1) $w^2 = u^2 + v^2 + 2uv\cos\alpha$.

The direction of w is inclined to that of u at an angle θ given by

$$\tan\theta = v\sin\alpha/(u + v\cos\alpha).$$

(2) $w^2 = \Sigma u^2 + 2\Sigma u_1 u_2 \cos \angle u_1 u_2$.

(3) See *STAT.* § 2 (4)–(8), all of which theorems hold *mutatis mutandis*.

3. Accelerations. The results of §§ 1, 2 above are true in every case if for the word 'velocity' we substitute

'acceleration', and for the words 'is at rest' the words 'has no actual acceleration'.

4. Relative Velocity.

(1) The velocities of two bodies relative to one another are unaltered by impressing on both any common velocity.

(2) The velocity of A relative to B is the velocity compounded of the velocity of A and a velocity equal and *opposite* to that of B.

(3) The same laws hold for *Relative Accelerations.*

5. Angular Velocity.

(1) If a point describes a circle of radius a with uniform velocity v and angular velocity ω about the centre, $v = a\omega$.

Note.—For a wheel rolling uniformly along the ground without slipping, if the velocity of the centre be v, that of the highest point is $2v$, that of the lowest point 0.

(2) If the velocity of a point whose polar coordinates are (r, θ) is at any instant v, its angular velocity about the pole at that instant is given by $r\omega = v \sin \phi$, where ϕ is the $\angle$ between r and the tangent at (r, θ) to the curve described by the point. This is also written $r^2\omega = vp$.

6. (1) *Uniform Motion*: $s = ut$.

(2) *Uniformly Accelerated Motion*:

$$\begin{aligned} v &= u + ft, \\ s &= ut + \tfrac{1}{2}ft^2, \\ v^2 &= u^2 + 2fs. \end{aligned}$$

Notes.—(1) The space described *during* the nth second is

$$s = u + \tfrac{1}{2}f(2n-1).$$

(2) For *Retardations* f is negative.

(3) $v = \dfrac{ds}{dt}$; $f = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ for variable accelerations, &c.

(3) *Uniformly Accelerated Motion under Gravity.* For f write $g = 32$ (F.P.S. system), or 981 (C.G.S. system).

Notes.—(1) It is of the utmost importance to pay special attention to the directions of velocities and accelerations. Thus, if a body is projected *upwards* and this is chosen as the positive direction, g is negative and the formulae become $v = u - gt$, &c.

(2) The more accurate values of g are 32·19, 981·2 (London). It varies according to position on the earth from 32·09 to 32·25 (978 to 983). Its value in latitude λ and at a height of h metres above the sea is (Helmert) $980{\cdot}632 - 2{\cdot}593 \cos 2\lambda - 0{\cdot}007 \cos^2 2\lambda - 0{\cdot}0003086\, h$.

7. Motion on Smooth Inclined Plane. For f write $g \sin \alpha$ in the formulae of § 6 (2), where α is the inclination of the plane to the horizon, and see § 6 (3), *Note* (1).

Notes.—(1) This gives motion in the line of greatest slope and assumes that the plane is fixed and not free to move.

(2) The result of § 18 (3) is often useful.

Chords of a Circle. The time of sliding down all chords of a vertical circle which pass through either the highest or the lowest points of the circle is the same.

Notes.—(1) The time of descent is $t = \sqrt{2\,a/g}$, where a is the radius of the circle.

(2) The theorem and Note (1) are true if we substitute 'sphere' for 'circle', and the theorem, but not Note (1), is also true if the plane of the circle is inclined to the vertical.

(3) The theorem is a particular case of the following: If O is *any* point on the upper half of a vertical circle the time of descent from O to the circle is the same for all chords through O. The theorem is also true if the chords are rough and if the motion takes place in a medium whose resistance varies as the velocity.

Chords of Quickest Descent (*Brachistochrones*).

(1) To find the chord of quickest descent, PQ, from a point P to a curve, describe a circle with P as its highest point, and touching the curve.

(2) To find the chord of quickest descent, PQ, from a curve to a point P, describe a circle with P as its lowest point, and touching the curve.

(3) The chord of quickest descent from one curve to

another bisects at each end the ∠ between the normal to the curve and the vertical.

Note.—These constructions may give the line of *Slowest Descent*.

8. The Laws of Motion (*Newton*).

Law i. Every body continues in its state of rest or of uniform motion in a straight line except in so far as it be compelled by impressed force to change that state.

Law ii. The rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which that force acts.

Law iii. To every action there is an equal and opposite reaction.

9. Measurement of Force.

(1) $P = Mf$, $W = Mg$.

Units:—

M in lb., f in F.S. units, P in poundals.

or M in grams, f in C.S. units, P in dynes.

Note.—A poundal is roughly the weight of $\frac{1}{2}$ oz., a dyne is roughly the weight of a milligram. These are *absolute* units. The weight of a pound (= 32·19 poundals) or of a gram (= 981 dynes) is sometimes taken (*gravitation units*). The dyne is so small that the *megadyne* (10^6 dynes) is often employed.

(2) Since $P = Mf$, the *Parallelogram of Forces* (*STAT.* § 1) follows from the *Parallelogram of Accelerations* (*DYN.* § 3).

Note.—This is the most strictly logical proof of the Parallelogram of Forces. All other proofs (Duchayla's, &c.) depend on so-called axioms.

(3) *Astronomical Unit of Mass* is that mass which would attract an equal mass at unit distance with the unit of force.

Notes.—(1) Its value is approximately $1{\cdot}543 \times 10^7$ grams.

(2) Its dimensions are $L^3 T^{-2}$, *assuming that the gravitation constant* (λ of Note 3), *of whose nature we know nothing, is of zero dimensions.*

(3) Two small spherical masses m_1, m_2 (in grams), placed at a distance d cm. apart, attract one another with a force $\lambda m_1 m_2/d^2$ dynes, where $\lambda = 6{\cdot}66 \times 10^{-8}$.

(4) The mean density of the earth is 5·53.

10. Particular Cases of Motion.

(1) *Railway Trains.* The frictional resistance offered to railway trains depends on the velocity, and the retardation produced is not therefore constant, so that no exact solution to such problems can usually be found by elementary methods. Problems involving the work (see § 12) done by the engine in maintaining a uniform speed on the level or on a gradient are capable of solution. The law of resistance for the engine is not the same as for the train, but an approximate law is, for an ordinary passenger train at the usual speeds, that the frictional resistance in pounds weight per ton is given by $\frac{16}{3}+\frac{1}{256}V^2$, where V is the velocity in m. p. h. Wind may double this resistance, especially cross-winds which increase the flange-friction.

(2) *Masses m_1, m_2 connected by string over smooth pulley ($m_1 > m_2$).*

$$f = \frac{m_1 - m_2}{m_1 + m_2} g; \quad T = \frac{2 m_1 m_2}{m_1 + m_2} g \text{ (poundals, dynes).}$$

Note.—In applying these results to Atwood's machine there are many errors :

(*a*) *Rigidity of String* (not serious, if silk thread is used).

(*b*) *Weight of String* (often important, allowed for in one form of machine).

(*c*) *Friction of Air* (velocities are too small for this to be important).

(*d*) *Friction of Pulley* (can be approximately allowed for, if assumed uniform, by a counterweight which just overcomes it).

(*e*) *Mass of Pulley* (the work of the falling weights is partly taken up in energy of rotation of the wheelwork. To allow for this take $f = (m_1 - m_2)g/(m_1 + m_2 + \mu)$, where μ is a constant experimentally determined for each machine. μ will generally lie between $\frac{1}{2}M$ and M where M is the mass of the pulley, and is accurately equal to Mk^2/a^2 where k is the 'radius of gyration' and a is the radius of the pulley).

(3) *Mass m_1 on smooth table pulled by m_2 hanging vertically.*

$$f = \frac{m_2 g}{m_1 + m_2}; \quad T = \frac{m_1 m_2}{m_1 + m_2} g.$$

11. Impulse [= Force × Time during which it acts].

(1) Change of momentum of a particle in a given time is equal to impulse of force which produces it and is in the same direction.

(2) An 'impulsive force' is measured by its 'impulse', i. e. by the change of momentum produced.

Note.—An 'impulsive force' is an infinitely great force acting for an infinitely short time. See § 16 (3) Note (4).

12. Work and Energy.

(1) The measure of the *work* done by a force = (Force) × (Distance through which the point of application is moved in the direction of the force)

$$= P \,.\, d\cos\alpha = (P\cos\alpha)\,.\,d,$$

where d = actual displacement of point of application of P, and $\alpha = \angle Pd$.

(2) *Power* = rate of doing work.

(3) *Units.*

	F.P.S.	C.G.S.
Work	Foot-poundal	Erg [= centimetre-dyne]
Practical Unit	Foot-pound [32 ft.-poundals]	Joule = 10^7 ergs = $\frac{3}{4}$ ft.-pound nearly
Power	Foot-poundal-second	[Erg-second]
Practical Unit	Horse-power = 550 foot-pound-seconds = 746 watts	Watt = 1 joule-second = 10^7 erg-seconds Kilowatt = 1000 watts

Notes.—(1) The C.G.S. horse-power is defined as 75 kilogram-metre-seconds, and is about 735 watts, its precise equivalent depending on the value taken for g.

(2) The actual working power of a good horse is about $\frac{4}{5}$ H.P., and the maximum working power of an average man is about $\frac{1}{8}$ H.P. The H.P. of a locomotive is 600–1000, and the H.P. of the *Mauretania* is 68,000.

(3) *Determination of H.P.* A rope passed once completely round fly-wheel (radius r) of engine carries at one end small weight w, at the other is attached to spring balance fixed to floor, reading W. If fly-wheel can turn n times per second the H.P. at which engine is working is $2\pi nr(W-w)/550$.

(4) The work done in stretching an elastic string = (Mean of Tensions) × (Extension).

(5) *Kinetic Energy* of body of mass m moving with velocity v is $\frac{1}{2}mv^2$.

Potential Energy of body of mass m at height h above the earth's surface (regarded as surface of zero potential) $= mgh$.

13. Certain General Laws.

(1) *Motion of Centre of Inertia.* If particles m_1, m_2 ... move with velocities (in same direction) u_1, u_2 ..., and with accelerations f_1, f_2 ..., the Velocity of Centre of Inertia in that direction is $\Sigma mu/\Sigma m$, and its acceleration is $\Sigma mf/\Sigma m$.

(2) The motion of the C.I. of a system of particles is unaffected by any mutual action (collisions, attraction, &c.) between the particles.

(3) The motion of the C.I. of a system of particles is the same as if all the masses were collected at the C.I. and all the external forces applied there parallel to their original directions.

(4) *Law of Conservation of Momentum.* If the sum of the external forces acting on any system, resolved in any given direction, always vanishes, the total momentum of the system in that direction remains the same throughout the motion.

(5) *Law of Conservation of Energy.* If a body or system of bodies is in motion under a conservative system of forces (i.e. forces depending only on position or configuration of system and not on velocity or direction of motion) the sum of its kinetic and potential energies is constant.

Note.—Non-conservative forces are Friction, Resistance of Air, Impulses of Collision, &c.

(6) *D'Alembert's Principle.* If a force is applied at each part of a moving system equal to the mass of that part

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Note.—Non-conservative forces are Friction, Resistance of Air, Impulses of Collision, &c.

(6) *D'Alembert's Principle.* If a force is applied at each part of a moving system equal to the mass of that part

multiplied by its acceleration, and in the direction of that acceleration *reversed*, these forces form an equilibrating system with the external impressed forces which act.

(7) *D'Alembert's Principle* (Impulses). If an impulse (or simultaneous impulses) act upon a system of bodies, then this impulse is in statical equilibrium with the impulses formed by multiplying the mass of each part by its velocity *before* the action of the impulse together with the impulses formed by multiplying the mass of each part by its *reversed* velocity *after* the action.

14. Dimensions of Units in Mass (1, 0, 0), Length (0, 1, 0), Time (0, 0, 1): Angular Velocity (0, 0, − 1), Velocity (0, 1, − 1), Acceleration (0, 1, − 2), Volume Density (1, − 3, 0).

Force (1, 1, − 2), Momentum or Impulse (1, 1, − 1).

Work or Energy (1, 2, − 2), Power (1, 2, − 3).

15. Projectiles.

(1) The *trajectory* is a parabola with axis vertical.

(2) The *resultant velocity* at any point on the trajectory is that due to a vertical fall from the directrix to the point.

Note.—So that the vertical height of the directrix above the point of projection and the distance of the focus from the point of projection are both $u^2/2g$.

(3) Motion in any direction is independent of motion in a perpendicular direction.

Hence.—*Vertical Velocity* after time t is $u \sin \alpha - gt$.

Horizontal Velocity after time t is $u \cos \alpha$.

Inclination of motion to horizon is $\tan \theta = (u \sin \alpha - gt)/u \cos \alpha$.

Vertical Height after time t is $h = u \sin \alpha \,.\, t - \frac{1}{2} gt^2$.

Horizontal Distance after time t is $d = u \cos \alpha \,.\, t$.

(4) *Time of Flight*: $2u \sin \alpha/g$.

Range on Horizontal Plane: $u^2 \sin 2\alpha/g$.

Proof.—$h = 0$, $\therefore\ t = 0$ or $2u \sin \alpha/g$. But $d = u \cos \alpha \,.\, t$.

Maximum Range ($\alpha = 45°$): u^2/g. [See (9) below.]

(5) *Range on Inclined Plane* (Angle $+\beta$):

$$2u^2 \cos\alpha \sin(\alpha-\beta)/g\cos^2\beta.$$

Proof.—For t consider motion perpendicular to plane. Range = (hor. dist. in t) $\times \sec\beta$.

Maximum Range. The angle of projection bisects the angle between the plane and the vertical (whether β be + or −).

(6) *Greatest Height*: $u^2\sin^2\alpha/2g$.

Proof.—Vert. vel. $= 0$, $\therefore\ t = u\sin\alpha/g$. Hence h.

(7) *Latus Rectum of Path*: $2u^2\cos^2\alpha/g$.

Equation to Path: $y = x\tan\alpha - (g/2u^2\cos^2\alpha)\,x^2$.

(8) *Envelope of Paths* (u const., α variable) is a parabola with focus at the point of projection and common directrix of the paths for tangent at vertex.

Proof.—Geometrical or by second result of (7) β.

(9) *Application to Guns*, &c. With comparatively low velocities, heavy shot, and high angles of fire the parabolic theory gives good approximate results. Thus with $v = 98$ metres p. s. and $\alpha = 38°$ a range of 928 m. was obtained (*Enc. Math.* iv. 3, p. 202) which is 98 % of the parabolic range. With higher velocities, lighter shot, and lower angles the parabolic theory is worthless. Thus for German infantry musket with $v = 640$ m. p. s., $\alpha = 4°$, the actual range of 1,612 m. is only 28 % of the parabolic range (ref. cit., *supra*). The resistance of the air follows no simple law. It is proportional, roughly, to the square of the velocity for low (< 750 f. p. s.) and high ($> 1{,}300$) velocities, but is proportional to the cube or fifth power for intermediate velocities (ref. cit., p. 195). Its general effect is to make the path much steeper in descent than in ascent, and the final velocity much less than the initial, and to diminish the time of flight. Thus for $v = 640$ m. p. s., $\alpha = 6°\ 28'{\cdot}3$, the range was 2,000 m. (parabolic theory 9,346), the angle of fall 13° 45′, and the final velocity 159 (ref. cit., p. 216). The actual maximum range is stated (W. M. Roberts, *Dynamics*, p. 230) to be attained at 33°, but this figure must depend on the velocity, and according to Greenhill (*Encyc. Brit.* iii. 275) the longest recorded range is 12 miles [$v = 2{,}375$ f. p. s., $\alpha = 40°$, time of flight 64″]. According to the parabolic theory the range is 33 miles, and the time of flight 95″. The parabolic theory probably gives fair results for cricket-balls, but experiment on this subject is to be desired and would interest schoolboys.

16. Collision.

(1) *Newton's Law*: (*a*) If two bodies impinge directly, the

relative velocity after impact $= -e$ (relative velocity before impact).

(*b*) If two bodies impinge obliquely, the relative normal velocity after impact $= -e$ (relative normal velocity before impact).

Notes.—(1) e = *Coefficient of Restitution* (*Elasticity*), depends only on substances of bodies, is independent of masses. Values (Hodgkinson), 0·94 (glass, glass), 0·81 (ivory, ivory), 0·20 (lead, lead), 0·13 (cast iron, lead).

(2) Fair results may be obtained by dropping balls and noting height to which they rise [$e = \sqrt{h'/h}$]. The heights may be fixed by rings of a retort stand.

(3) The law is only approximate. The ratio of relative velocities decreases slightly as the velocities increase.

(2) *Direct Collision.* The results of the collision of two *smooth* spheres, or of a smooth sphere against a smooth fixed surface, may be obtained by *Newton's Law* and the principle of the *Conservation of Momentum* [§ 13 (4)]. The following are added for reference:

α. *Particle on Fixed Smooth Plane*, velocities before and after impact u and v making angles α, θ with the normal:

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha),$$

$$\cot\theta = e\cot\alpha.$$

$$\text{Loss of energy} = \tfrac{1}{2}(1-e^2)u^2\cos^2\alpha.$$

β. *Masses m, m' moving along same line* with velocities u, u':

$$(m+m')v = (m-em')u + m'(1+e)u',$$

$$(m+m')v' = m(1+e)u + (m'-em)u'.$$

$$\text{Loss of energy} = \tfrac{1}{2}(1-e^2)(u-u')^2 mm'/(m+m').$$

Note that if $m = m'$, $e = 1$, the particles exchange velocities on collision.

(3) *Collision of Constrained Bodies.* The impact must be divided into two parts, a 'period of compression' at the end of which the colliding bodies will have zero relative *normal* velocity, and a 'period of restitution' at the end of which they are just separating. If I and I' be the (normal) impulses during these periods, $I' = eI$.

Notes.—(1) This law, which can be proved true for the cases considered in (2), is *assumed* as the real interpretation of Newton's Law for all cases, and is sometimes given as Newton's Law.

(2) It is important for the beginner to notice that it is *only the velocities along the normal* which are the same at the end of the period of compression. The velocities must not be equated along any other line of resolution. Further, it is *only* the velocities *at the end of the period of compression*, and not at the end of the period of restitution.

(3) The *Law of Conservation of Momentum* [§ 13 (4)] is always applicable. That of *Conservation of Energy* is NOT. Some problems which really belong to *Rigid Dynamics* may require D'Alembert's Principle [§ 13 (7)]. Others may be indeterminate.

(4) The time of contact of two steel spheres, rad. 2·5 cm., meeting with relative velocity 1 cm. p. s. has been shown (Hertz) to be 0·00038 sec., the radius of the surface of contact 0·013 cm., and the maximum total pressure 2·47 kilograms.

17. The Hodograph.

(1) If a particle P is moving in any manner, and if from any fixed point O a line OQ be drawn $\parallel$ and $::^{l}$ to the velocity of P at any moment, the locus of Q is the *hodograph* of P.

(2) The velocity of Q represents in magnitude and direction the acceleration of P.

(3) *Special Hodographs.*

(*a*) P describes a st. line with uniform acceleration; Q describes a parallel line with uniform velocity.

(*b*) P describes a circle with uniform velocity; Q also describes a circle with uniform velocity.

(*c*) P a projectile; Q describes a vertical st. line with uniform velocity.

Proof.—*Either* directly from definition (velocity at any pt. = velocity at P compounded with gt downwards), *or* $v^2 = 2g \cdot SP \propto SY^2$, $\therefore v \propto SY$. But Y lies on tangent at vertex.

(*d*) P describes a central orbit; Q describes the polar reciprocal [*G. CON.* § 35] of the orbit turned through a right angle.

Proof.—For vp = constant.

(*e*) P describes a S. H. M.; Q describes an equal S. H. M. with phase a quarter period in advance.

18. Motion in a Circle or Curve.

(1) A point moving uniformly in a circle of radius r has a normal (i. e. central) acceleration $v^2/r = r\omega^2$ inwards.

(2) If a point is moving in a curve and its velocity at any moment be v, its normal acceleration at that moment is v^2/ρ inwards, where ρ is the radius of curvature at the point.

(3) If a point slides down the arc of any smooth curve or moves under gravity under constraints always at right angles to the direction of motion, and v and u be its velocity at points of which the former is a vertical height h below the latter,

$$v^2 = u^2 + 2gh.$$

Note.—As examples of 'constraints' a ball swinging freely attached to a string, a particle moving in any path on the surface of a smooth sphere, or in a smooth tube, &c.

(4) *Conical Pendulum.* A body tied to a point by a string of length l, and moving so that the string remains stretched and sweeps out a right circular cone with axis vertical and vertical angle 2α, will rotate once round the axis in time

$$t = 2\pi\sqrt{l\cos\alpha/g}.$$

19. Simple Harmonic Motion.

(1) If P describes a circle, centre O, radius a, uniformly with angular velocity ω, and PN be $\perp^r$ from P to a diameter, N describes a S. H. M.

Displacement: $x = ON = a\cos\omega t$.

Velocity of $N = dx/dt = -a\omega\sin\omega t$.

Acceleration of $N = dv/dt = -a\omega^2\cos\omega t = -\omega^2 x$.

Periodic Time $= 2\pi/\omega$.

Notes.—(1) The above assumes that $t = 0$ when $x = a$. More generally $x = a\cos(\omega t + \epsilon)$ where ϵ is the *epoch*.

(2) The composition of two S. H. M.'s in the same direction is effected by $x = a\cos(\omega t + \epsilon) + a'\cos(\omega' t + \epsilon')$, and if $\omega = \omega'$ (i. e. if the periods are the same) the resultant is a S. H. M.

(3) The composition of two S. H. M.'s in directions at right angles

gives rise to *Lissajou's Figures*, whose equations are found by eliminating t between $x = a\cos(\omega t+\epsilon)$. $y = b\cos(\omega' t+\epsilon')$.

(4) The essential conditions for a force to produce S.H.M. are (a) the force must be continuously in the direction of the displacement, (b) it must be proportional to the displacement and in the opposite direction. These are satisfied, approximately, by the great majority of forces resisting displacement in nature. If a body distant x from a position of equilibrium (measured along a straight path or a curved path) be continually acted on by a force, per unit of mass, of $-\lambda x$ in the tangent to the path it will oscillate in a S.H.M. of period $2\pi/\sqrt{\lambda}$, since $\lambda = \omega^2$. It is assumed throughout this *Note* that the displacements are small.

(2) *Pendulum.* The periodic time of a simple pendulum is $2\pi\sqrt{l/g}$ and is approximately independent of the amplitude of swing (**Isochronism**).

Notes.—(1) More exactly, if 2α be the total angle of swing,

$$t = 2\pi\sqrt{l/g}\,(1+\tfrac{1}{4}\sin^2\tfrac{1}{2}\alpha).$$

So that the error in t, through lack of isochronism, is 0·05 % ($\alpha = 5°$), 0·2 % ($\alpha = 10°$), 0·4 % ($\alpha = 15°$).

The exact formula is

$$t = 2\pi\sqrt{l/g}\left[1+\frac{1^2}{2^2}\sin^2\frac{1}{2}\alpha+\frac{1^2\cdot 2^2}{3^2\cdot 4^2}\sin^4\frac{1}{2}\alpha+\ldots\right].$$

(2) If the pendulum be composed of a uniform sphere of radius r hanging by a string of length l (measured to the centre of the sphere), for l we must substitute in the above formula $l+\frac{2}{5}(r^2/l)$.

(3) *Cycloidal Motion.* [For Geometry of the Cycloid see *DIFF. CALC.* § 28 (3).] If a smooth cycloid is placed with axis vertical and vertex downwards, the time of oscillation of a particle starting to slide downwards from *any* point of the cycloid is $2\pi\sqrt{4a/g}$, where a is the radius of the generating circle.

(4) *Cycloidal Pendulum.* A pendulum becomes strictly isochronous if the supporting thread be attached to the cusp of a metal cam in the shape of a complete horizontal cycloid (vertices downward), and initially pulled taut along one branch of this cycloid.

Note.—Even the resistance of the air does not sensibly affect this isochronism, though it would cause the oscillations to diminish in amplitude.

20. Central Orbits.

(1) For any orbit described by P wholly under forces tending to a fixed point O, if v be the velocity of P in the orbit, ω the angular velocity round O, p the perpendicular from O upon the line of v, $r = OP$,

$$vp = \text{constant } (h) = r^2\omega.$$

(2) OP describes equal areas in equal times.

(3) *Special Cases.*

(*a*) *Force* $= \mu r$ (per unit mass).

Orbit: Ellipse with O as centre (C).

Periodic Time: $2\pi/\sqrt{\mu}$.

Velocity: $\sqrt{\mu}\,.\,CD$.

Proof.—Resolve force into μx, μy and use § 19 (1).

(*b*) *Force* $= \mu/r^2$ (per unit mass).

Orbit: Conic with O as focus.

Periodic Time: $2\pi\sqrt{a^3/\mu}$, if orbit is elliptic.

Velocity: $v^2 = \mu(2/r + 1/a)$ if hyperbolic,

$= \mu(2/r)$ if parabolic,

$= \mu(2/r - 1/a)$ if elliptic.

Note.—Hence orbit is hyperbolic, parabolic, or elliptic, as square of velocity of projection $>$, $=$, or $< 2\mu/r$.

(4) *Kepler's Law of Planetary Motion.*

i. The planets describe ellipses having the sun in one focus.

Note.—The actual fact is that the earth (for instance) and the sun both describe ellipses round their common C. Gr. as focus, but as this C. Gr. is situated only about 280 miles from the sun's centre the law may be taken as true.

ii. The radius vector from the sun to any planet describes equal areas in equal times.

iii. The squares of the periodic times of all the planets

are proportional to the cubes of the major axes of their orbits.

Notes.—(1) From Law ii we deduce [(2) above] that the planets are acted on by a force towards the sun, from Law i [by (3) (*b*)] that this force $= \mu/r^2$, from Law iii that μ is the same for all planets.

(2) The laws hold also if for 'planet' we read 'satellite', and for 'sun' we read 'planet'.

(6) The pressure on the base of a vessel containing liquid depends only on the area of the base and the depth of the liquid and is independent of the slope or shape of the sides.

3. The Thrusts.

[Horizontal (H.T.), Vertical (V.T.), Resultant (R.T.).]

Name.	*Magnitude.*	*Point of Action.*
R.T. on planes.	$swhS$; S is area of plane, h depth C.Gr.	C.P. of plane. [See § 4.]
V.T. on surfaces.	Wt. of liquid vertically above S.	C.Gr. of this liquid.
V.T. on immersed body.	Wt. of liquid displaced.	C.Gr. of this liquid.
H.T. on surface.	H.T. on S in any direction is R.T. upon projection of S upon a vertical plane perpendicular to that direction.	C.P. of projection. [See § 4.]
R.T. on surface.	For planes: see above.	
	For other surfaces: resultant of V.T. and H.T.	Intersection of V.T. and H.T.

Notes.—(1) The phrase 'Whole Pressure' unfortunately occurs still. Its value is $swhS$. But *it has no meaning whatever* unless S is plane, and then it is the same as Resultant Thrust.

(2) The V.T. may act downwards (liquid *above* S, as when S is part of the base of a bottle filled with water), or upwards (liquid below S, as when S is part of the shoulder of the bottle).

(3) The V.T. for immersed bodies is found by same rule whether body is wholly or partially immersed.

(4) H.T. The projection should be taken parallel to a (vertical) plane of symmetry of S. If there is no such plane the H.T. in two directions at right angles must be taken and their resultant found.

(5) The R.T. on a curved surface may be found more easily if the surface is bounded by a plane curve, from the fact that R.T. on surface and R.T. on plane bounded by curve are statically equivalent to weight of liquid between surface and plane (see V.T. on immersed body).

(6) This consideration will often give the C.P. in a difficult case, e.g. for C.P. of a circle wholly immersed (plane not necessarily

vertical) consider a hemisphere bounded by circle, R. T. on hemisphere must pass through centre of sphere; take moments about centre to express equivalence of R. T. on circle and R. T. on hemisphere (moment zero) to wt. of hemisphere of liquid.

(7) Hence we can deal with pressure on (*a*) curved surface of right circular cylinder with plane ends, (*b*) curved surface of a right circular cone, (*c*) curved surface of frustum of cone with plane circular ends.

4. Centre of Pressure (C. P.).

Methods of Determination.

i. C. P. of any plane area (*S*) is the point of the area vertically below the C. Gr. of the solid formed by drawing vertical lines through the boundary of the area to meet the surface of the liquid.

ii. To determine C. P. of any plane area draw horizontal lines through the boundary of the area, each line being in length ::1 to the pressure at the boundary-point through which it is drawn; determine C. Gr. of solid thus formed; a horizontal line through this C. Gr. meets the area in its C. P.

iii. See § 3, *Note* (6).

iv. If a line of symmetry Ox of S meets the surface of the liquid in O and meets S in A and B, and if S be so placed that the ordinates of S at right angles to Ox are all horizontal, the distance of the C. P. from O along Ox is

$$\left[\int_{OA}^{OB} x^2 y\,dx\right] \Big/ \left[\int_{OA}^{OB} xy\,dx\right].$$

v. Split into rectangles and triangles and see below.

Particular Cases.

(1) *Rectangle*, one edge in surface: C. P. $\frac{2}{3}$ down median.

(2) *Triangle*, one side in surface: C. P. $\frac{1}{2}$ down median.

(3) *Triangle*, vertex in surface, opposite side horizontal: C. P. $\frac{3}{4}$ down median.

Note.—With a little practice these are instantaneously obtained mentally by Method i, and there is danger in memorizing them.

(4) *Triangle*, anyhow immersed: C. P. is centre of ∥ forces acting at mid-points of sides and of magnitudes ::[1] to depths.

Note.—If α, β, γ be depths of angular points, C. P. is at depth

$$\tfrac{1}{2}[\Sigma\alpha^2+\Sigma\beta\gamma]/\Sigma\alpha = \tfrac{1}{2}\Sigma a - \frac{1}{6h}\Sigma\alpha\beta,$$

where h is depth of its C. Gr.

(5) *Quadrilaterals, Polygons*: Method v, above.

Note.—If α, β, γ, δ be depths of vertices of a 4[1] and h the depth of its C. Gr., the depth of its C. P. is $\frac{1}{2}\Sigma\alpha - \frac{1}{6h}\Sigma\alpha\beta$.

(6) *Circle*, wholly immersed (plane vertical, depth of centre h, radius a). Depth of C. P. $h+a^2/4h$. [§ 3, *Note* (6).]

Note.—If the plane of the ⊙ be inclined at $\angle\,\alpha$ to horizon the C. P. is $\frac{1}{4}(a^2/h)\sin\alpha$ from the centre, measured along the median line downwards, h being still the *depth* of the centre. The two results may be learnt, if desired, as one: If d be the *distance* of the centre from the surface measured along the median line, then the distance of the C. P. is always $d+a^2/4d$.

5. Equilibrium of Floating Bodies.

(1) The forces acting are:

(α) Upward pressure of liquid displaced through C. Gr. of this liquid (**Centre of Buoyancy**).

(β) Weight of body, downwards, through its C. Gr.

(γ) Any external impressed forces such as tension of string, reaction at hinge, &c.

Apply ordinary statical methods to these.

(2) A body floating freely is in equilibrium if $\alpha = \beta$ [§ 5 (1)] and if the point of action of α is in the same vertical line as the point of action of β.

(3) The equilibrium of a body floating freely is *stable* or *unstable* according as the *Metacentre* (§ 6) is above or below the centre of gravity. If the immersed portion is spherical the metacentre is the centre of the sphere. For other cases see § 6.

Notes.—(1) If the metacentre and the centre of gravity coincide the equilibrium is *neutral.*

(2) For submarines the condition of stability is that the C. Gr. should be below the C.B.

6. METACENTRE.

(1) *Centre of Buoyancy* (C. B.) is the C. Gr. of the liquid displaced.

Surface of Buoyancy: If a floating body is displaced so that the volume of the part immersed remains constant, the locus (in the body) of the C. B. is the surface of buoyancy.

Curve of Buoyancy is the locus of the C. B. for displacements such that the C. B. remains in one plane.

(2) The tangent plane at any point to the surface of buoyancy is ∥ to the plane of floatation (P. F.).

(3) The *Positions of Equilibrium* are found by drawing normals from the C. Gr. to the surface of buoyancy and taking P. F. $\perp^r$ to these normals.

(4) The *Metacentre* for planar displacements of the C. B. is the centre of curvature of the curve of buoyancy at the C. B. for the position considered.

(5) *Position of Metacentre.* If the body is symmetrical and if the displacements be about an axis of symmetry in the P. F., the distance of the metacentre above the C. B. = I/V, where I = 'Moment of Inertia' of the P. F. about this axis of symmetry, V = volume of the fluid displaced.

Special cases:

P. F. a circle: $I = \pi a^4/4$.

P. F. a rectangle (sides a, b): $I = a^3 b/12$, the displacements being in a plane at right angles to edge b.

Lamina (floating vertically): $I = a^3/12$, where a = length of section by liquid and V = area immersed.

Practical Determination of Metacentre.

The 'Metacentric Height' (i.e. height of metacentre above the *centre of gravity*) is worked out before the ship is built from the working

diagrams, but as the position of the C. Gr., which depends on the loads, can only be roughly ascertained, a further 'inclining experiment' is made when the ship is completed by moving weight w a distance h from one side of the deck to the other. This causes the ship to tilt through an angle θ, which is determined by a plumb-line, and it is quite easy to prove that if W be the weight ('displacement') of the ship, the metacentric height is $hw \cot\theta/W$. The actual metacentric heights are greatest in shallow-draught river gunboats, in which they vary from 8 to 20 feet, and least in large mail steamers, in which they vary from 6 in. to 2 feet. For men-of-war they are between $3\frac{1}{2}$ and 5 feet. A small value diminishes the safety, a large value causes excessive and rapid rolling (*Encyc. Brit.* xxiv. 924).

(6) *Curves of Buoyancy.*

Triangle (*Vertex Immersed*): A hyperbola with immersed sides as asymptotes.

Rectangle: A parabola with axis the vertical line of symmetry of the rectangle, concavity upwards, and tangent at the vertex the horizontal through the C. B.

Notes.—(1) For rolling displacement of an ordinary ship the C. B. is approximately an arc of a hyperbola, but more nearly a parabolic arc if the sides are almost perpendicular to the water.

(2) It is an admirable exercise to trace out completely the buoyancy curves and their evolutes (the metacentric locus) in these cases. In the case of the rectangle the buoyancy curve consists of a closed curve made up of four parabolic arcs united by four arcs of rectangular hyperbolas, which latter vanish if the rectangle floats with exactly half of it immersed.

7. Rotating Liquids.

(1) If a liquid, whether confined or free, rotate about a vertical axis with angular velocity ω, the surfaces of equal pressure are equal paraboloids of revolution, whose axes are the axis of revolution and whose equation is

$$x^2 = (2g/\omega^2)\, y.$$

(2) The free surface in such a case takes the form of a paraboloid of revolution, and the pressure at any point is the pressure up to the free surface.

(3) To find the pressure at any point in a confined rotating liquid:

(a) If the liquid does not quite fill the containing vessel

the position of the free surface must be determined by the consideration that the paraboloid generated by $x^2 = (2g/\omega^2)\,y$ has for volume the volume of space left empty in the vessel. This gives the free surface of zero pressure, and thus the pressure at any point.

(*b*) If the liquid just fills the vessel, the surface of zero pressure is the paraboloid with axis the axis of revolution and touching the rotating vessel at the point of zero pressure.

8. Surface Tension.

Cylinder. If the surface tension (perpendicular to length of cylinder) at any point of a thin long cylinder (rad. r), where the hydrostatic pressure is p, be t per unit of length, $t = pr$.

Sphere. If the surface tension at any point of a thin sphere (rad. r) be t in any direction, $2t = pr$.

Note.—If the cylinder be of finite uniform thickness, c and T be the tension per unit area of the substance, $t = cT$.

9. Gases.

(1) *Boyle's Law.* The pressure of a given mass of gas varies inversely as its volume, provided that the temperature is unaltered, i.e. pv = constant, if t is constant.

(2) *Charles's Law*: $v_t = v_0\,(1 + \alpha t)$.

(p constant, $\alpha = \frac{1}{273}$ nearly.)

(3) *Deductions*:

(α) $\rho_t = \rho_0/(1 + \alpha t)$. ($p$ constant, $\alpha = \frac{1}{273}$ nearly.)

(β) pv/T = constant. (T = absolute temperature.)

(γ) p/ρ is always the same for a given gas (t constant).

(4) *Height by Barometer.* The difference of height at two stations at which the barometric readings are h_1, h is, approximately,

$$1{\cdot}84 \log_{10} [h_1/h] \times 10^4 \text{ metres.}$$

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Five-figure Logarithmic and Trigonometrical Tables

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These five-figure tables are intended to give results correct to four figures; the fifth figure in the answer may be inaccurate.

The decimal point is printed before all the logarithms of numbers; it is hoped that this will obviate the common mistake of reading off logarithms instead of antilogarithms, and vice-versa.

The trigonometrical tables are arranged so that, at one opening of the tables, all the functions of an angle may be found on the left-hand page and their logarithms on the right-hand page; here again confusion is avoided. The characteristics of the logarithmic functions are the true characteristics; no useful purpose is served by increasing them by 10.

It should be noticed that, instead of dividing by a sine, one may multiply by the cosecant, &c., and, similarly, instead of subtracting the logarithm of a sine, one may add the logarithm of the cosecant, &c. In many cases this shortens calculation.

For quick reference the last page may be used, which gives the trigonometrical functions, to four figures only, for every whole degree up to 90° and the corresponding circular measure to five figures.

Logarithms of R for Compound Interest

R	log R	R	log R
1·0025	·00108438	1·0325	·01389006
1·005	·00216606	1·035	·01494035
1·0075	·00324505	1·0375	·01598811
1·01	·00432137	1·04	·01703334
1·0125	·00539503	1·0425	·01807606
1·015	·00646604	1·045	·01911629
1·0175	·00753442	1·0475	·02015403
1·02	·00860017	1·05	·02118930
1·0225	·00966332	1·0525	·02222210
1·025	·01072387	1·055	·02325246
1·0275	·01178183	1·0575	·02428038
1·03	·01283722	1·06	·02530587

Constants used in Mensuration and their Logarithms

	logarithm		logarithm
$\pi = 3{\cdot}14159265$	0·497150	$1 \div \pi = 0{\cdot}31830989$	$\bar{1}{\cdot}502850$
$\frac{1}{2}\pi = 1{\cdot}57079633$	0·196120	$1 \div 4\pi = 0{\cdot}07957747$	$\bar{2}{\cdot}900790$
$\frac{1}{6}\pi = 0{\cdot}52359878$	$\bar{1}{\cdot}718999$	$\sqrt[3]{6 \div \pi} = 1{\cdot}24070098$	0·093667
$\frac{4}{3}\pi = 4{\cdot}18879020$	0·622089	$\sqrt[3]{3 \div 4\pi} = 0{\cdot}62035049$	$\bar{1}{\cdot}792637$
$\sqrt{\pi} = 1{\cdot}77245385$	0·248575	$\sqrt{1 \div \pi} = 0{\cdot}56418958$	$\bar{1}{\cdot}751425$
$\pi^2 = 9{\cdot}86960440$	0·994300	$1 \div \pi^2 = 0{\cdot}10132118$	$\bar{1}{\cdot}005700$
$\sqrt[3]{\pi} = 1{\cdot}46459189$	0·165717	$\sqrt[3]{\pi^2} = 2{\cdot}14502940$	0·331433
$\pi/180 = 0{\cdot}01745329$	$\bar{2}{\cdot}241877$	$180/\pi = 57{\cdot}29577951$	1·758123

Naperian (or Natural) Logarithms

$e = 2{\cdot}7182182 \quad \log_{10} e = {\cdot}43429448 \quad \log_e 10 = 2{\cdot}30258509$

$\log_{10} N = \log_e N \times \log_{10} e. \qquad \log_e N = \log_{10} N \times \log_e 10$

	0	1	2	3	4	Mean Differences 1	2	3	4	5	6	7	8	9
10	·00000	·00432	·00860	·01284	·01703	42	85	127	170	212	254	297	339	381
11	·04139	·04532	·04922	·05308	·05690	39	77	116	155	193	232	270	309	348
12	·07918	·08279	·08636	·08991	·09342	35	71	106	142	177	213	248	284	319
13	·11394	·11727	·12057	·12385	·12710	33	66	98	131	164	197	229	262	295
14	·14613	·14922	·15229	·15534	·15836	30	61	91	122	152	183	213	244	274
15	·17609	·17898	·18184	·18469	·18752	28	57	85	114	142	171	199	228	256
16	·20412	·20683	·20952	·21219	·21484	27	53	80	107	134	160	187	214	240
17	·23045	·23300	·23553	·23805	·24055	25	50	76	101	126	151	176	201	227
18	·25527	·25768	·26007	·26245	·26482	24	48	71	95	119	143	167	190	214
19	·27875	·28103	·28330	·28556	·28780	23	45	68	90	113	135	158	180	203
20	·30103	·30320	·30535	·30750	·30963	21	43	64	86	107	128	150	172	193
21	·32222	·32428	·32634	·32838	·33041	20	41	61	82	102	123	143	164	184
22	·34242	·34439	·34635	·34830	·35025	20	39	59	78	98	117	137	158	176
23	·36173	·36361	·36549	·36736	·36922	19	37	56	75	94	112	131	149	168
24	·38021	·38202	·38382	·38561	·38739	18	36	54	72	90	108	125	143	161
25	·39794	·39967	·40140	·40312	·40483	17	34	52	69	86	103	120	138	155
26	·41497	·41664	·41830	·41996	·42160	17	33	50	66	83	99	116	132	149
27	·43136	·43297	·43457	·43616	·43775	16	32	48	64	80	96	112	128	143
28	·44716	·44871	·45025	·45179	·45332	15	31	46	61	77	92	108	123	138
29	·46240	·46389	·46538	·46687	·46835	15	30	45	59	74	89	104	119	134
30	·47712	·47857	·48001	·48144	·48287	14	29	43	57	72	86	101	115	129
31	·49136	·49276	·49415	·49554	·49693	14	28	42	56	70	83	97	111	125
32	·50515	·50651	·50786	·50920	·51055	13	27	40	54	67	81	94	108	121
33	·51851	·51983	·52114	·52244	·52375	13	26	39	52	65	78	91	105	118
34	·53148	·53275	·53403	·53529	·53656	13	25	38	51	63	76	89	101	114
35	·54407	·54531	·54654	·54777	·54900	12	25	37	49	62	74	86	99	111
36	·55630	·55751	·55871	·55991	·56110	12	24	36	48	60	72	84	96	108
37	·56820	·56937	·57054	·57171	·57287	12	23	35	47	58	70	82	93	105
38	·57978	·58093	·58206	·58320	·58433	11	23	34	45	57	68	80	91	102
39	·59106	·59218	·59329	·59439	·59550	11	22	33	44	55	66	78	89	100
40	·60206	·60314	·60423	·60531	·60638	11	22	32	43	54	65	76	86	97
41	·61278	·61384	·61490	·61595	·61700	11	21	32	42	53	63	74	84	95
42	·62325	·62428	·62531	·62634	·62737	10	21	31	41	51	62	72	82	93
43	·63347	·63448	·63548	·63649	·63749	10	20	30	40	50	60	70	80	90
44	·64345	·64444	·64542	·64640	·64738	10	20	29	39	49	59	69	79	88
45	·65321	·65418	·65514	·65610	·65706	10	19	29	38	48	58	67	77	86
46	·66276	·66370	·66464	·66558	·66652	9	19	28	38	47	56	66	75	84
47	·67210	·67302	·67394	·67486	·67578	9	18	28	37	46	55	64	73	83
48	·68124	·68215	·68305	·68395	·68485	9	18	27	36	45	54	63	72	81
49	·69020	·69108	·69197	·69285	·69373	9	18	26	35	44	53	62	71	79
50	·69897	·69984	·70070	·70157	·70243	9	17	26	35	43	52	60	69	78
51	·70757	·70842	·70927	·71012	·71096	8	17	25	34	42	51	59	68	76
52	·71600	·71684	·71767	·71850	·71933	8	17	25	33	42	50	58	67	75
53	·72428	·72509	·72591	·72673	·72754	8	16	24	33	41	49	57	65	73
54	·73239	·73320	·73400	·73480	·73560	8	16	24	32	40	48	56	64	72

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·02119	·02531	·02938	·03342	·03743	40	81	121	162	202	242	283	323	364
11	·06070	·06446	·06819	·07188	·07555	37	74	111	148	185	222	259	296	333
12	·09691	·10037	·10380	·10721	·11059	34	68	102	137	170	204	238	272	307
13	·13033	·13354	·13672	·13988	·14301	32	63	95	126	158	190	221	253	284
14	·16137	·16435	·16732	·17026	·17319	29	59	88	118	147	177	206	236	265
15	·19033	·19312	·19590	·19866	·20140	28	55	83	110	138	165	193	221	248
16	·21748	·22011	·22272	·22531	·22789	26	52	78	104	130	156	182	208	233
17	·24304	·24551	·24797	·25042	·25285	24	49	73	98	123	147	171	196	220
18	·26717	·26951	·27184	·27416	·27646	23	46	70	93	116	139	162	185	208
19	·29003	·29226	·29447	·29667	·29885	22	44	66	88	110	132	154	176	198
20	·31175	·31387	·31597	·31806	·32015	21	42	63	84	105	126	147	168	188
21	·33244	·33445	·33646	·33846	·34044	20	40	60	80	100	120	140	160	180
22	·35218	·35411	·35603	·35793	·35984	19	38	57	77	96	115	134	153	172
23	·37107	·37291	·37475	·37658	·37840	18	37	55	73	91	110	128	146	165
24	·38917	·39094	·39270	·39445	·39620	18	35	53	70	88	105	123	140	158
25	·40654	·40824	·40993	·41162	·41330	17	34	51	67	84	101	118	135	152
26	·42325	·42488	·42651	·42813	·42975	16	32	49	65	81	97	114	130	146
27	·43933	·44091	·44248	·44404	·44560	16	31	47	63	78	94	110	125	141
28	·45484	·45637	·45788	·45939	·46090	15	30	45	61	76	91	106	121	136
29	·46982	·47129	·47276	·47422	·47567	14	29	44	58	73	87	102	117	131
30	·48430	·48572	·48714	·48855	·48996	14	28	42	56	71	85	99	113	127
31	·49831	·49969	·50106	·50243	·50379	14	27	41	55	68	82	96	109	123
32	·51188	·51322	·51455	·51587	·51720	13	27	40	53	66	80	93	106	119
33	·52504	·52634	·52763	·52892	·53020	13	26	39	51	64	77	90	103	116
34	·53782	·53908	·54033	·54158	·54283	13	25	38	50	63	75	88	100	113
35	·55023	·55145	·55267	·55388	·55509	12	24	36	49	61	73	85	97	109
36	·56229	·56348	·56467	·56585	·56703	12	24	35	47	59	71	83	95	106
37	·57403	·57519	·57634	·57749	·57864	12	23	35	46	58	69	81	92	104
38	·58546	·58659	·58771	·58883	·58995	11	22	34	45	56	67	78	90	101
39	·59660	·59770	·59879	·59988	·60097	11	22	33	44	55	66	76	87	98
40	·60746	·60853	·60959	·61066	·61172	11	21	32	43	53	64	74	85	96
41	·61805	·61909	·62014	·62118	·62221	10	21	31	42	52	62	73	83	94
42	·62839	·62941	·63043	·63144	·63246	10	20	30	41	51	61	71	81	91
43	·63849	·63949	·64048	·64147	·64246	10	20	30	40	50	60	70	79	89
44	·64836	·64933	·65031	·65128	·65225	10	19	29	39	49	58	68	78	87
45	·65801	·65986	·65992	·66087	·66181	10	19	29	38	48	57	67	76	86
46	·66745	·66839	·66932	·67025	·67117	9	19	28	37	47	56	65	74	84
47	·67669	·67761	·67852	·67943	·68034	9	18	27	36	46	55	64	73	82
48	·68574	·68664	·68753	·68842	·68931	9	18	27	36	45	54	62	71	80
49	·69461	·69548	·69636	·69723	·69810	9	17	26	35	44	52	61	70	78
50	·70329	·70415	·70501	·70586	·70672	9	17	26	34	43	51	60	68	77
51	·71181	·71265	·71349	·71433	·71517	8	17	25	34	42	50	59	67	75
52	·72016	·72099	·72181	·72263	·72346	8	16	25	33	41	49	58	66	74
53	·72835	·72916	·72997	·73078	·73159	8	16	24	32	40	48	57	65	73
54	·73640	·73719	·73799	·73878	·73957	8	16	24	32	40	48	55	63	71

Mean Differences

	0	1	2	3	4	1	2	3	4	5	6	7	8	9
55	·74036	·74115	·74194	·74273	·74351	8	16	24	31	39	47	55	63	71
56	·74819	·74896	·74974	·75051	·75128	8	15	23	31	39	46	54	62	69
57	·75587	·75664	·75740	·75815	·75891	8	15	23	30	38	46	53	61	68
58	·76343	·76418	·76492	·76567	·76641	7	15	22	30	37	45	52	60	67
59	·77085	·77159	·77232	·77305	·77379	7	15	22	29	37	44	51	59	66
60	·77815	·77887	·77960	·78032	·78104	7	14	22	29	36	43	51	58	65
61	·78533	·78604	·78675	·78746	·78817	7	14	21	28	36	43	50	57	64
62	·79239	·79309	·79379	·79449	·79518	7	14	21	28	35	42	49	56	63
63	·79934	·80003	·80072	·80140	·80209	7	14	21	27	34	41	48	55	62
64	·80618	·80686	·80754	·80821	·80889	7	14	20	27	34	41	47	54	61
65	·81291	·81358	·81425	·81491	·81558	7	13	20	27	33	40	47	53	60
66	·81954	·82020	·82086	·82151	·82217	7	13	20	26	33	39	46	52	59
67	·82607	·82672	·82737	·82802	·82866	6	13	19	26	32	39	45	52	58
68	·83251	·83315	·83378	·83442	·83506	6	13	19	25	32	38	45	51	57
69	·83885	·83948	·84011	·84073	·84136	6	13	19	25	31	38	44	50	56
70	·84510	·84572	·84634	·84696	·84757	6	12	19	25	31	37	43	49	56
71	·85126	·85187	·85248	·85309	·85370	6	12	18	24	31	37	43	49	55
72	·85733	·85794	·85854	·85914	·85974	6	12	18	24	30	36	42	48	54
73	·86332	·86392	·86451	·86510	·86570	6	12	18	24	30	36	42	48	53
74	·86923	·86982	·87040	·87099	·87157	6	12	18	23	29	35	41	47	53
75	·87506	·87564	·87622	·87679	·87737	6	11	17	23	29	35	40	46	52
76	·88081	·88138	·88195	·88252	·88309	6	11	17	23	29	34	40	46	51
77	·88649	·88705	·88762	·88818	·88874	6	11	17	22	28	34	39	45	51
78	·89209	·89265	·89321	·89376	·89432	6	11	17	22	28	33	39	44	50
79	·89763	·89818	·89873	·89927	·89982	5	11	16	22	27	33	38	44	49
80	·90309	·90363	·90417	·90472	·90526	5	11	16	22	27	33	38	43	49
81	·90849	·90902	·90956	·91009	·91062	5	11	16	21	27	32	37	43	48
82	·91381	·91434	·91487	·91540	·91593	5	11	16	21	27	32	37	42	48
83	·91908	·91960	·92012	·92065	·92117	5	10	16	21	26	31	37	42	47
84	·92428	·92480	·92531	·92583	·92634	5	10	15	21	26	31	36	41	46
85	·92942	·92993	·93044	·93095	·93146	5	10	15	20	26	31	36	41	46
86	·93450	·93500	·93551	·93601	·93651	5	10	15	20	25	30	35	40	45
87	·93952	·94002	·94052	·94101	·94151	5	10	15	20	25	30	35	40	45
88	·94448	·94498	·94547	·94596	·94645	5	10	15	20	25	30	34	39	44
89	·94939	·94988	·95036	·95085	·95134	5	10	15	19	24	29	34	39	44
90	·95424	·95472	·95521	·95569	·95617	5	10	14	19	24	29	34	39	43
91	·95904	·95952	·99599	·96047	·96095	5	10	14	19	24	29	33	38	43
92	·96379	·96426	·96473	·96520	·96567	5	9	14	19	24	28	33	38	42
93	·96848	·96895	·96942	·96988	·97035	5	9	14	19	23	28	33	37	42
94	·97313	·97359	·97405	·97451	·97497	5	9	14	18	23	28	32	37	41
95	·97772	·97818	·97864	·97909	·97955	5	9	14	18	23	27	32	36	41
96	·98227	·98272	·98318	·98363	·98408	5	9	14	18	23	27	32	36	41
97	·98677	·98722	·98767	·98811	·98856	4	9	13	18	22	27	31	36	40
98	·99123	·99167	·99211	·99255	·99300	4	9	13	18	22	27	31	35	40
99	·99564	·99607	·99651	·99695	·99739	4	9	13	17	22	26	31	35	39

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·74429	·74507	·74586	·74663	·74741	8	16	23	31	39	47	55	62	70
56	·75205	·75282	·75358	·75435	·75511	8	15	23	31	38	46	53	61	69
57	·75967	·76042	·76118	·76193	·76268	8	15	23	30	38	45	53	60	68
58	·76716	·76790	·76864	·76938	·77012	7	15	22	30	37	44	52	59	66
59	·77452	·77525	·77597	·77670	·77743	7	15	22	29	36	44	51	58	65
60	·78176	·78247	·78319	·78390	·78462	7	14	21	29	36	43	50	57	64
61	·78888	·78958	·79029	·79099	·79169	7	14	21	28	35	42	49	56	63
62	·79588	·79657	·79727	·79796	·79865	7	14	21	28	35	42	48	55	62
63	·80277	·80346	·80414	·80482	·80550	7	14	20	27	34	41	48	55	61
64	·80956	·81023	·81090	·81158	·81224	7	13	20	27	34	40	47	54	60
65	·81624	·81690	·81757	·81823	·81889	7	13	20	26	33	40	46	53	59
66	·82282	·82347	·82413	·82478	·82543	7	13	20	26	33	39	46	52	59
67	·82930	·82995	·83059	·83123	·83187	6	13	19	26	32	39	45	51	58
68	·83569	·83632	·83696	·83759	·83822	6	13	19	25	32	38	44	51	57
69	·84198	·84261	·84323	·84386	·84448	6	12	19	25	31	37	44	50	56
70	·84819	·84880	·84942	·85003	·85065	6	12	18	25	31	37	43	49	55
71	·85431	·85491	·85552	·85612	·85673	6	12	18	24	30	36	42	48	54
72	·86034	·86094	·86153	·86213	·86273	6	12	18	24	30	36	42	48	54
73	·86629	·86688	·86747	·86806	·86864	6	12	18	24	29	35	41	47	53
74	·87216	·87274	·87332	·87390	·87448	6	12	17	23	29	35	41	46	52
75	·87795	·87852	·87910	·87967	·88024	6	11	17	23	29	34	40	46	51
76	·88366	·88423	·88480	·88536	·88593	6	11	17	23	28	34	40	45	51
77	·88930	·88986	·89042	·89098	·89154	6	11	17	22	28	33	39	45	50
78	·89487	·89542	·89597	·89653	·89708	6	11	17	22	28	33	39	44	50
79	·90037	·90091	·90146	·90200	·90255	5	11	16	22	27	33	38	44	49
80	·90580	·90634	·90687	·90741	·90795	5	11	16	22	27	32	38	43	48
81	·91116	·91169	·91222	·91275	·91328	5	11	16	21	27	32	37	42	48
82	·91646	·91698	·91751	·91803	·91855	5	10	16	21	26	31	37	42	47
83	·92169	·92221	·92273	·92324	·92376	5	10	16	21	26	31	36	41	47
84	·92686	·92737	·92788	·92840	·92891	5	10	15	20	26	31	36	41	46
85	·93197	·93247	·93298	·93349	·93399	5	10	15	20	25	30	35	40	46
86	·93702	·93752	·93802	·93852	·93902	5	10	15	20	25	30	35	40	45
87	·94201	·94250	·94300	·94349	·94399	5	10	15	20	25	30	35	40	44
88	·94694	·94743	·94792	·94841	·94890	5	10	15	20	25	29	34	39	44
89	·95182	·95231	·95279	·95328	·95376	5	10	15	19	24	29	34	39	44
90	·95665	·95713	·95761	·95809	·95856	5	10	14	19	24	29	33	38	43
91	·96142	·96190	·96237	·96284	·96332	5	9	14	19	24	28	33	38	43
92	·96614	·96661	·96708	·96755	·96802	5	9	14	19	23	28	33	37	42
93	·97081	·97128	·97174	·97220	·97267	5	9	14	19	23	28	32	37	42
94	·97543	·97589	·97635	·97681	·97727	5	9	14	18	23	27	32	37	41
95	·98000	·98046	·98091	·98137	·98182	5	9	14	18	23	27	32	36	41
96	·98453	·98498	·98543	·98588	·98632	4	9	13	18	22	27	31	36	40
97	·98900	·98945	·98989	·99034	·99078	4	9	13	18	22	27	31	36	40
98	·99344	·99388	·99432	·99476	·99520	4	9	13	18	22	26	31	35	40
99	·99782	·99826	·99870	·99913	·99957	4	9	13	17	22	26	31	35	39

Mean Differences

ANTILOGARITHMS

	0	1	2	3	4	Mean Differences 1	2	3	4	5	6	7	8	9
·00	10000	10023	10046	10069	10093	2	5	7	9	12	14	16	19	21
·01	10233	10257	10280	10304	10328	2	5	7	9	12	14	17	19	21
·02	10471	10495	10520	10544	10568	2	5	7	10	12	15	17	20	22
·03	10715	10740	10765	10789	10814	2	5	7	10	12	15	17	20	22
·04	10965	10990	11015	11041	11066	3	5	8	10	13	15	18	20	23
·05	11220	11246	11272	11298	11324	3	5	8	10	13	16	18	21	23
·06	11482	11508	11535	11561	11588	3	5	8	11	13	16	18	21	24
·07	11749	11776	11803	11830	11858	3	5	8	11	14	16	19	22	24
·08	12023	12050	12078	12106	12134	3	6	8	11	14	17	19	22	25
·09	12303	12331	12359	12388	12417	3	6	9	11	14	17	20	23	26
·10	12589	12618	12647	12677	12706	3	6	9	12	15	18	20	23	26
·11	12882	12912	12942	12972	13002	3	6	9	12	15	18	20	24	27
·12	13183	13213	13243	13274	13305	3	6	9	12	15	18	21	24	27
·13	13490	13521	13552	13583	13614	3	6	9	12	16	19	22	25	28
·14	13804	13836	13868	13900	13932	3	6	10	13	16	19	22	26	29
·15	14125	14158	14191	14223	14256	3	7	10	13	16	20	23	26	30
·16	14454	14488	14521	14555	14588	3	7	10	13	17	20	24	27	30
·17	14791	14825	14859	14894	14928	3	7	10	14	17	21	24	27	31
·18	15136	15171	15205	15241	15276	4	7	11	14	18	21	25	28	32
·19	15488	15524	15560	15596	15631	4	7	11	14	18	22	25	29	32
·20	15849	15885	15922	15959	15996	4	7	11	15	18	22	26	29	33
·21	16218	16255	16293	16331	16368	4	8	11	15	19	23	26	30	34
·22	16596	16634	16672	16711	16749	4	8	11	15	19	23	27	31	34
·23	16982	17022	17061	17100	17140	4	8	12	16	20	24	28	32	35
·24	17378	17418	17458	17498	17539	4	8	12	16	20	24	28	32	36
·25	17783	17824	17865	17906	17947	4	8	12	16	21	25	29	33	37
·26	18197	18239	18281	18323	18365	4	8	13	17	21	25	30	34	38
·27	18621	18664	18707	18750	18793	4	9	13	17	22	26	30	34	39
·28	19055	19099	19143	19187	19231	4	9	13	18	22	26	31	35	40
·29	19498	19543	19588	19634	19679	5	9	14	18	23	27	32	36	41
·30	19953	19999	20045	20091	20137	5	9	14	18	23	28	32	37	42
·31	20417	20464	20512	20559	20606	5	9	14	19	24	28	33	38	43
·32	20893	20941	20989	21038	21086	5	10	15	19	24	29	34	39	44
·33	21380	21429	21478	21528	21577	5	10	15	20	25	30	35	40	44
·34	21878	21928	21979	22029	22080	5	10	15	20	25	30	35	40	46
·35	22387	22439	22491	22542	22594	5	10	16	21	26	31	36	41	47
·36	22909	22961	23014	23067	23121	5	11	16	21	27	32	37	42	48
·37	23442	23496	23550	23605	23659	5	11	16	22	27	33	38	44	49
·38	23988	24044	24099	24155	24210	6	11	17	22	28	33	39	44	50
·39	24547	24604	24660	24717	24774	6	11	17	23	28	34	40	45	51
·40	25119	25177	25235	25293	25351	6	12	17	23	29	35	41	47	52
·41	25704	25763	25823	25882	25942	6	12	18	24	30	36	42	48	54
·42	26303	26363	26424	26485	26546	6	12	18	24	30	36	43	49	55
·43	26915	26977	27040	27102	27164	6	12	19	25	31	37	44	50	56
·44	27542	27606	27669	27733	27797	6	13	19	26	32	38	45	51	57
·45	28184	28249	28314	28379	28445	7	13	20	26	33	39	46	52	59
·46	28840	28907	28973	29040	29107	7	13	20	27	33	40	47	53	60
·47	29512	29580	29648	29717	29785	7	14	21	27	34	41	48	55	62
·48	30200	30269	30339	30409	30479	7	14	21	28	35	42	49	56	63
·49	30903	30974	31046	31117	31189	7	14	21	29	36	43	50	57	64

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	10116	10139	10162	10186	10209	2	5	7	9	12	14	16	19	21
·01	10351	10375	10399	10423	10447	2	5	7	10	12	14	17	19	22
·02	10593	10617	10641	10666	10691	2	5	7	10	12	15	17	20	22
·03	10839	10864	10889	10914	10940	3	5	8	10	13	15	18	20	23
·04	11092	11117	11143	11169	11194	3	5	8	10	13	15	18	20	23
·05	11350	11376	11402	11429	11455	3	5	8	11	13	16	18	21	24
·06	11614	11641	11668	11695	11722	3	5	8	11	14	16	19	22	24
·07	11885	11912	11940	11967	11995	3	6	8	11	14	17	19	22	25
·08	12162	12190	12218	12246	12274	3	6	8	11	14	17	20	23	25
·09	12445	12474	12503	12531	12560	3	6	9	12	14	17	20	23	26
·10	12735	12764	12794	12823	12853	3	6	9	12	15	18	21	24	26
·11	13032	13062	13092	13122	13152	3	6	9	12	15	18	21	24	27
·12	13335	13366	13397	13428	13459	3	6	9	12	16	19	22	25	28
·13	13646	13677	13709	13740	13772	3	6	9	13	16	19	22	25	28
·14	13964	13996	14028	14060	14093	3	6	10	13	16	19	23	26	29
·15	14289	14322	14355	14388	14421	3	7	10	13	17	20	23	26	30
·16	14622	14655	14689	14723	14757	3	7	10	14	17	20	24	27	30
·17	14962	14997	15031	15066	15101	3	7	10	14	17	21	24	28	31
·18	15311	15346	15382	15417	15453	4	7	11	14	18	21	25	28	32
·19	15668	15704	15740	15776	15812	4	7	11	14	18	22	25	29	33
·20	16032	16069	16106	16144	16181	4	7	11	15	19	22	26	30	33
·21	16406	16444	16482	16520	16558	4	8	11	15	19	23	27	30	34
·22	16788	16827	16866	16904	16943	4	8	12	16	19	23	27	31	35
·23	17179	17219	17258	17298	17338	4	8	12	16	20	24	28	32	36
·24	17579	17620	17660	17701	17742	4	8	12	16	20	24	29	33	37
·25	17989	18030	18072	18113	18155	4	8	12	17	21	25	29	33	37
·26	18408	18450	18493	18535	18578	4	9	13	17	21	26	30	34	38
·27	18836	18880	18923	18967	19011	4	9	13	18	22	26	31	35	39
·28	19275	19320	19364	19409	19454	4	9	13	18	22	27	31	36	40
·29	19724	19770	19815	19861	19907	5	9	14	18	23	27	32	37	41
·30	20184	20230	20277	20324	20370	5	9	14	19	23	28	33	37	42
·31	20654	20701	20749	20797	20845	5	10	14	19	24	29	33	38	43
·32	21135	21184	21232	21281	21330	5	10	15	20	25	29	34	39	44
·33	21627	21677	21727	21777	21827	5	10	15	20	25	30	35	40	45
·34	22131	22182	22233	22284	22336	5	10	15	20	26	31	36	41	46
·35	22646	22699	22751	22803	22856	5	11	16	21	26	32	37	42	47
·36	23174	23227	23281	23335	23388	5	11	16	21	27	32	38	43	48
·37	23714	23768	23823	23878	23933	5	11	16	22	27	33	38	44	49
·38	24266	24322	24378	24434	24491	6	11	17	22	28	34	39	45	51
·39	24831	24889	24946	25003	25061	6	12	17	23	28	35	40	46	52
·40	25410	25468	25527	25586	25645	6	12	18	24	29	35	41	47	53
·41	26002	26062	26122	26182	26242	6	12	18	24	30	36	42	48	54
·42	26607	26669	26730	26792	26853	6	12	18	25	31	37	43	49	55
·43	27227	27290	27353	27416	27479	6	13	19	25	32	38	44	50	57
·44	27861	27925	27990	28054	28119	6	13	19	26	32	39	45	52	58
·45	28510	28576	28642	28708	28774	7	13	20	26	33	40	46	53	59
·46	29174	29242	29309	29376	29444	7	14	20	27	34	41	47	54	61
·47	29854	29923	29992	30061	30130	7	14	21	28	35	42	48	55	62
·48	30549	30620	30690	30761	30832	7	14	21	28	35	42	50	57	64
·49	31261	31333	31405	31477	31550	7	14	21	29	36	43	51	58	65

Mean Differences

	0	1	2	3	4	1	2	3	4	5	6	7	8	9
·50	31623	31696	31769	31842	31915	7	15	22	29	37	44	51	59	66
·51	32359	32434	32509	32584	32659	8	15	23	30	38	45	53	60	68
·52	33113	33189	33266	33343	33420	8	15	23	31	38	46	53	61	69
·53	33884	33963	34041	34119	34198	8	16	24	31	39	47	55	63	71
·54	34674	34754	34834	34914	34995	8	16	24	32	40	48	56	64	72
·55	35481	35563	35645	35727	35810	8	16	25	33	41	49	58	66	74
·56	36308	36392	36475	36559	36644	8	17	25	34	42	50	59	67	76
·57	37154	37239	37325	37411	37497	9	17	26	34	43	52	60	69	78
·58	38019	38107	38194	38282	38371	9	18	26	35	44	53	62	70	79
·59	38905	38994	39084	39174	39264	9	18	27	36	45	54	63	72	81
·60	39811	39902	39994	40087	40179	9	18	28	37	46	55	65	74	83
·61	40738	40832	40926	41020	41115	9	19	28	38	47	57	66	76	85
·62	41687	41783	41879	41976	42073	10	19	29	39	48	58	67	77	87
·63	42658	42756	42855	42954	43053	10	20	30	40	49	59	69	79	89
·64	43652	43752	43853	43954	44055	10	20	30	40	51	61	71	81	91
·65	44668	44771	44875	44978	45082	10	21	31	41	52	62	73	83	93
·66	45709	45814	45920	46026	46132	11	21	32	42	53	63	74	85	95
·67	46774	46881	46989	47098	47206	11	22	32	43	54	65	76	86	97
·68	47863	47973	48084	48195	48306	11	22	33	44	55	66	78	89	100
·69	48978	49091	49204	49317	49431	11	23	34	45	57	68	79	91	102
·70	50119	50234	50350	50466	50582	12	23	35	46	58	70	81	93	104
·71	51286	51404	51523	51642	51761	12	24	36	48	59	71	83	95	107
·72	52481	52602	52723	52845	52966	12	24	36	49	61	73	85	97	109
·73	53703	53827	53951	54075	54200	12	25	37	50	62	75	87	100	112
·74	54954	55081	55208	55335	55463	13	25	38	51	64	76	89	102	114
·75	56234	56364	56494	56624	56754	13	26	39	52	65	78	91	104	117
·76	57544	57677	57810	57943	58076	13	27	40	53	67	80	93	107	120
·77	58884	59020	59156	59293	59429	14	27	41	55	68	82	95	109	123
·78	60256	60395	60534	60674	60814	14	28	42	56	70	84	98	112	126
·79	61660	61802	61944	62087	62230	14	29	43	57	71	86	100	114	128
·80	63096	63241	63387	63533	63680	15	29	44	58	73	88	102	117	131
·81	64565	64714	64863	65013	65163	15	30	45	60	75	90	105	120	135
·82	66069	66222	66374	66527	66681	15	31	46	61	77	92	107	122	138
·83	67608	67764	67920	68077	68234	16	31	47	63	78	94	110	125	141
·84	69183	69343	69502	69663	69823	16	32	48	64	80	96	112	128	144
·85	70795	70958	71121	71285	71450	16	33	49	66	82	98	115	131	147
·86	72444	72611	72778	72946	73114	17	34	50	67	84	101	117	134	151
·87	74131	74302	74473	74645	74817	17	34	51	69	86	103	120	137	154
·88	75858	76033	76208	76384	76560	18	35	53	70	88	105	123	140	158
·89	77625	77804	77983	78163	78343	18	36	54	72	90	108	126	144	162
·90	79433	79616	79799	79983	80168	18	37	55	74	92	110	129	147	166
·91	81283	81470	81658	81846	82035	19	38	56	75	94	113	132	151	169
·92	83176	83368	83560	83753	83946	19	39	58	77	96	116	135	154	174
·93	85114	85310	85507	85704	85901	20	39	59	79	99	118	138	158	177
·94	87096	87297	87498	87700	87902	20	40	61	81	101	121	141	161	182
·95	89125	89331	89536	89743	89950	21	41	62	83	103	124	144	165	186
·96	91201	91411	91622	91833	92045	21	42	63	84	106	127	148	169	190
·97	93325	93541	93756	93972	94189	22	43	65	86	108	130	151	173	195
·98	95499	95719	95940	96161	96383	22	44	66	88	111	133	155	177	199
·99	97724	97949	98175	98401	98628	23	45	68	90	113	136	158	181	204

	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	31989	32063	32137	32211	32285	7	15	22	30	37	44	52	59	67
·51	32734	32810	32885	32961	33037	8	15	23	30	38	45	53	61	68
·52	33497	33574	33651	33729	33806	8	15	23	31	39	46	54	62	70
·53	34277	34356	34435	34514	34594	8	16	24	32	40	48	55	63	71
·54	35075	35156	35237	35318	35400	8	16	24	32	41	49	57	65	73
·55	35892	35975	36058	36141	36224	8	17	25	33	42	50	58	67	75
·56	36728	36813	36898	36983	37068	9	17	26	34	43	51	60	68	77
·57	37584	37670	37757	37844	37931	9	17	26	35	44	52	61	70	78
·58	38459	38548	38637	38726	38815	9	18	27	36	45	54	62	71	80
·59	39355	39446	39537	39628	39719	9	18	27	36	46	55	64	73	82
·60	40272	40365	40458	40551	40644	9	19	28	37	47	56	65	75	84
·61	41210	41305	41400	41495	41591	10	19	29	38	48	57	67	76	86
·62	42170	42267	42364	42462	42560	10	20	29	39	49	59	68	78	88
·63	43152	43251	43351	43451	43551	10	20	30	40	50	60	70	80	90
·64	44157	44259	44361	44463	44566	10	20	31	41	51	61	72	82	92
·65	45186	45290	45394	45499	45604	10	21	31	42	52	63	73	84	94
·66	46238	46345	46452	46559	46666	11	21	32	43	54	64	75	86	96
·67	47315	47424	47534	47643	47753	11	22	33	44	55	66	77	88	99
·68	48417	48529	48641	48753	48865	11	22	34	45	56	67	79	90	101
·69	49545	49659	49774	49888	50003	11	23	34	46	57	69	80	92	103
·70	50699	50816	50933	51051	51168	12	23	35	47	59	70	82	94	106
·71	51880	52000	52119	52240	52360	12	24	36	48	60	72	84	96	108
·72	53088	53211	53333	53456	53580	12	25	37	49	62	74	86	98	111
·73	54325	54450	54576	54702	54828	13	25	38	50	63	75	88	100	113
·74	55590	55719	55847	55976	56105	13	26	39	52	64	77	90	103	116
·75	56885	57016	57148	57280	57412	13	26	40	53	66	79	92	105	119
·76	58210	58345	58479	58614	58749	13	27	40	54	67	81	94	108	121
·77	59566	59704	59841	59979	60117	14	28	41	55	69	82	97	110	124
·78	60954	61094	61235	61376	61518	14	28	42	56	71	85	99	113	127
·79	62373	62517	62661	62806	62951	14	29	43	58	72	87	101	116	130
·80	63826	63973	64121	64269	64417	15	30	44	59	74	88	104	118	133
·81	65313	65464	65615	65766	65917	15	30	45	60	76	91	106	121	136
·82	66834	66988	67143	67298	67453	15	31	46	62	77	93	108	124	139
·83	68391	68549	68707	68865	69024	16	32	48	63	79	95	111	127	143
·84	69984	70146	70307	70469	70632	16	32	49	65	81	97	114	130	146
·85	71614	71779	71945	72111	72277	17	33	50	66	83	100	116	133	149
·86	73282	73451	73621	73790	73961	17	34	51	68	84	102	119	136	153
·87	74989	75162	75336	75509	75683	17	35	52	70	87	104	122	139	156
·88	76736	76913	77090	77268	77446	18	36	53	71	89	107	124	142	160
·89	78524	78705	78886	79068	79250	18	36	55	73	91	109	127	145	164
·90	80353	80538	80724	80910	81096	19	37	56	74	93	112	130	149	167
·91	82224	82414	82604	82794	82985	19	38	57	76	95	114	133	152	171
·92	84140	84333	84528	84723	84918	19	39	58	78	97	117	136	156	175
·93	86099	86298	86497	86696	86896	20	40	60	80	100	120	140	160	179
·94	88105	88308	88512	88716	88920	20	41	61	82	102	122	143	163	184
·95	90157	90365	90573	90782	90991	21	42	63	84	104	125	146	167	188
·96	92257	92470	92683	92897	93111	21	43	64	85	107	128	150	171	192
·97	94406	94624	94842	95060	95280	22	44	66	87	109	131	153	175	197
·98	96605	96828	97051	97275	97499	22	45	67	90	112	134	157	179	201
·99	98855	99083	99312	99541	99770	23	46	69	92	115	137	160	183	206

Mean Differences

Differences are given for every 10′. Intermediate values can be found by th method of proportional parts; e. g.:—

To find tan 43° 56′ and cos 37° 34′

tan 43° 50′ = ·96008	cos 37° 30′ = ·79335
+ diff. for 6′ = 337	− diff. for 4′ = − 71
∴ tan 43° 56′ = ·96345	∴ cos 37° 34′ = ·79264

When there is no entry in the difference column, the value of the function change too rapidly for correct interpolation by proportional parts. Greater accuracy is the obtained by expressing the function in terms of the sine and cosine.

To find tan 67° 23′

tan 67° 20′ = 2·39449 Diff. for 10′ = 1972

by proportional parts, diff. for 3′ = 592

This gives tan 67° 23′ = 2·40041. (The correct value is 2·40038.)

Subtract differences when dealing with co-functions

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
0°	·00000	291	∞		·00000	291	∞		1·00000	000	1·00000	000	90°
10′	·00291	291	343·78		·00291	291	343·77		1·00000	002	1·00000	002	50′
20′	·00582	291	171·89		·00582	291	171·89		1·00002	002	·99998	002	40′
30′	·00873	291	114·59		·00873	291	114·59		1·00004	003	·99996	003	30′
40′	·01164	290	85·946		·01164	291	85·940		1·00007	004	·99993	004	20′
50′	·01454	291	68·757		·01455	291	68·750		1·00011	004	·99989	004	10′
1°	·01745	291	57·299		·01746	291	57·290		1·00015	005	·99985	006	89°
10′	·02036	291	49·114		·02037	291	49·104		1·00021	006	·99979	006	50′
20′	·02327	291	42·976		·02328	291	42·964		1·00027	007	·99973	007	40′
30′	·02618	290	38·202		·02619	291	38·188		1·00034	008	·99966	008	30′
40′	·02908	291	34·382		·02910	291	34·368		1·00042	009	·99958	009	20′
50′	·03199	291	31·258		·03201	291	31·242		1·00051	010	·99949	010	10′
2°	·03490	291	28·654		·03492	291	28·636		1·00061	011	·99939	010	88°
10′	·03781	290	26·451		·03783	292	26·432		1·00072	011	·99929	012	50′
20′	·04071	291	24·562		·04075	291	24·542		1·00083	012	·99917	012	40′
30′	·04362	291	22·926		·04366	292	22·904		1·00095	013	·99905	013	30′
40′	·04653	290	21·494		·04658	291	21·470		1·00108	014	·99892	014	20′
50′	·04943	291	20·230		·04949	292	20·206		1·00122	015	·99878	015	10′
3°	·05234		19·107		·05241		19·081		1·00137		·99863		87°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

87°

LOGARITHMIC FUNCTIONS

The values given here are the true logarithms; the characteristic is not increased by 10 as in many tables.

Differences are given for every 10′. Intermediate values can be found by the method of proportional parts.

The differences for the logarithm of a function and of the reciprocal of the function are the same in magnitude but opposite in sign.

When there is no entry in the difference column, the rate of change of the logarithm changes too rapidly for correct interpolation by proportional parts.

The following rules may be used when the angle is small:—

Log sine. Add $\bar{6}{\cdot}68557$ to the log of the angle expressed in seconds and subtract $\frac{1}{3}$ of the log secant.

Log tan. Add $\bar{6}{\cdot}68557$ to the log of the angle expressed in seconds and add $\frac{2}{3}$ of the log secant.

When the log sine is given, the angle is found in seconds by adding $5{\cdot}31443$ to the log sine and $\frac{1}{3}$ of the corresponding log secant (found in the ordinary way).

When the log tan is given, the angle is found in seconds by adding $5{\cdot}31443$ to the log tan and subtracting $\frac{2}{3}$ of the corresponding log secant (found in the ordinary way).

Subtract differences when dealing with co-functions

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
0°	— ∞		∞	— ∞		∞	0·00000		0·00000	90°
10′	$\bar{3}{\cdot}46373$		2·53627	$\bar{3}{\cdot}46373$		2·53627	0·00000	001	0·00000	50′
20′	$\bar{3}{\cdot}76475$		2·23525	$\bar{3}{\cdot}76476$		2·23524	0·00001	001	$\bar{1}{\cdot}99999$	40′
30′	$\bar{3}{\cdot}94084$		2·05916	$\bar{3}{\cdot}94086$		2·05914	0·00002	001	$\bar{1}{\cdot}99998$	30′
40′	$\bar{2}{\cdot}06578$		1·93422	$\bar{2}{\cdot}06581$		1·93419	0·00003	002	$\bar{1}{\cdot}99997$	20′
50′	$\bar{2}{\cdot}16268$		1·83732	$\bar{2}{\cdot}16273$		1·83727	0·00005	002	$\bar{1}{\cdot}99995$	10′
1°	$\bar{2}{\cdot}24186$		1·75814	$\bar{2}{\cdot}24192$		1·75808	0·00007	002	$\bar{1}{\cdot}99993$	89°
10′	$\bar{2}{\cdot}30879$		1·69121	$\bar{2}{\cdot}30888$		1·69112	0·00009	003	$\bar{1}{\cdot}99991$	50′
20′	$\bar{2}{\cdot}36678$		1·63322	$\bar{2}{\cdot}36689$		1·63311	0·00012	003	$\bar{1}{\cdot}99988$	40′
30′	$\bar{2}{\cdot}41792$		1·58208	$\bar{2}{\cdot}41807$		1·58193	0·00015	003	$\bar{1}{\cdot}99985$	30′
40′	$\bar{2}{\cdot}46366$		1·53634	$\bar{2}{\cdot}46385$		1·53615	0·00018	004	$\bar{1}{\cdot}99982$	20′
50′	$\bar{2}{\cdot}50504$		1·49496	$\bar{2}{\cdot}50527$		1·49473	0·00022	004	$\bar{1}{\cdot}99978$	10′
2°	$\bar{2}{\cdot}54282$		1·45718	$\bar{2}{\cdot}54308$		1·45692	0·00026	004	$\bar{1}{\cdot}99974$	88°
10′	$\bar{2}{\cdot}57757$		1·42243	$\bar{2}{\cdot}57788$		1·42212	0·00031	005	$\bar{1}{\cdot}99969$	50′
20′	$\bar{2}{\cdot}60973$		1·39027	$\bar{2}{\cdot}61009$		1·38991	0·00036	005	$\bar{1}{\cdot}99964$	40′
30′	$\bar{2}{\cdot}63968$		1·36032	$\bar{2}{\cdot}64009$		1·35991	0·00041	006	$\bar{1}{\cdot}99959$	30′
40′	$\bar{2}{\cdot}66769$		1·33231	$\bar{2}{\cdot}66816$		1·33184	0·00047	006	$\bar{1}{\cdot}99953$	20′
50′	$\bar{2}{\cdot}69400$		1·30600	$\bar{2}{\cdot}69453$		1·30547	0·00053	006	$\bar{1}{\cdot}99947$	10′
3°	$\bar{2}{\cdot}71880$		1·28120	$\bar{2}{\cdot}71940$		1·28060	0·00060		$\bar{1}{\cdot}99940$	87°
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

87°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
3°	·05234	290	19·1073		·05241	292	19·0811		1·00137	016	·99863	016	87°
10′	·05524	290	18·1026		·05533	291	18·0750		1·00153	016	·99847	016	50′
20′	·05814	291	17·1984		·05824	292	17·1693		1·00169	018	·99831	018	40′
30′	·06105	290	16·3804		·06116	292	16·3499		1·00187	018	·99813	018	30′
40′	·06395	290	15·6368		·06408	292	15·6048		1·00205	019	·99795	019	20′
50′	·06685	291	14·9579		·06700	293	14·9244		1·00224	020	·99776	020	10′
4°	·06976	290	14·3356		·06993	292	14·3007		1·00244	021	·99756	020	86°
10′	·07266	290	13·7631		·07285	293	13·7267		1·00265	022	·99736	022	50′
20′	·07556	290	13·2347		·07578	292	13·1969		1·00287	022	·99714	022	40′
30′	·07846	290	12·7455		·07870	293	12·7062		1·00309	024	·99692	024	30′
40′	·08136	290	12·2913		·08163	293	12·2505		1·00333	024	·99668	024	20′
50′	·08426	290	11·8684		·08456	293	11·8262		1·00357	025	·99644	025	10′
5°	·08716	289	11·4737		·08749	293	11·4301		1·00382	026	·99619	025	85°
10′	·09005	290	11·1046		·09042	293	11·0594		1·00408	027	·99594	027	50′
20′	·09295	290	10·7585		·09335	294	10·7119		1·00435	028	·99567	027	40′
30′	·09585	289	10·4334		·09629	294	10·3854		1·00463	028	·99540	029	30′
40′	·09874	290	10·1275		·09923	293	10·0780		1·00491	030	·99511	029	20′
50′	·10164	289	9·83912		·10216	294	9·78817		1·00521	030	·99482	030	10′
6°	·10453	289	9·56677		·10510	295	9·51436		1·00551	031	·99452	031	84°
10′	·10742	289	9·30917		·10805	294	9·25530		1·00582	032	·99421	031	50′
20′	·11031	289	9·06515		·11099	295	9·00983		1·00614	033	·99390	033	40′
30′	·11320	289	8·83367		·11394	294	8·77689		1·00647	034	·99357	033	30′
40′	·11609	289	8·61379		·11688	295	8·55554		1·00681	034	·99324	034	20′
50′	·11898	289	8·40466		·11983	295	8·34496		1·00715	036	·99290	035	10′
7°	·12187	289	8·20551		·12278	296	8·14435		1·00751	037	·99255	036	83°
10′	·12476	288	8·01565		·12574	295	7·95302		1·00788	037	·99219	037	50′
20′	·12764	288	7·83443		·12869	296	7·77035		1·00825	038	·99182	038	40′
30′	·13052	289	7·66130		·13165	296	7·59575		1·00863	039	·99144	038	30′
40′	·13341	288	7·49571		·13461	297	7·42871		1·00902	040	·99106	039	20′
50′	·13629	288	7·33719		·13758	296	7·26873		1·00942	041	·99067	040	10′
8°	·13917	288	7·18530		·14054	297	7·11537		1·00983	041	·99027	041	82°
10′	·14205	288	7·03962		·14351	297	6·96823		1·01024	043	·98986	042	50′
20′	·14493	288	6·89979		·14648	297	6·82694		1·01067	044	·98944	042	40′
30′	·14781	288	6·76547		·14945	298	6·69116		1·01111	044	·98902	044	30′
40′	·15069	287	6·63633		·15243	297	6·56055		1·01155	045	·98858	044	20′
50′	·15356	287	6·51208		·15540	298	6·43484		1·01200	047	·98814	045	10′
9°	·15643	288	6·39245		·15838	299	6·31375		1·01247	047	·98769	046	81°
10′	·15931	287	6·27719		·16137	298	6·19703		1·01294	048	·98723	047	50′
20′	·16218	287	6·16607		·16435	299	6·08444		1·01342	049	·98676	047	40′
30′	·16505	287	6·05886		·16734	299	5·97576		1·01391	049	·98629	049	30′
40′	·16792	286	5·95536		·17033	300	5·87080		1·01440	051	·98580	049	20′
50′	·17078	287	5·85539		·17333	300	5·76937		1·01491	052	·98531	050	10′
10°	·17365		5·75877		·17633		5·67128		1·01543		·98481		80°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
3°	**$\bar{2}$·71880**		**1·28120**	**$\bar{2}$·71940**		**1·28060**	**0·00060**	006	**$\bar{1}$·99940**	**87°**
10′	$\bar{2}$·74226		1·25774	$\bar{2}$·74292		1·25708	0·00066	008	$\bar{1}$·99934	50′
20′	$\bar{2}$·76541		1·23549	$\bar{2}$·76525		1·23475	0·00074	007	$\bar{1}$·99926	40′
30′	$\bar{2}$·78568		1·21432	$\bar{2}$·78649		1·21351	0·00081	008	$\bar{1}$·99919	30′
40′	$\bar{2}$·80585		1·19415	$\bar{2}$·80674		1·19326	0·00089	008	$\bar{1}$·99911	20′
50′	$\bar{2}$·82513		1·17487	$\bar{2}$·82610		1·17390	0·00097	009	$\bar{1}$·99903	10′
4°	**$\bar{2}$·84358**		**1·15642**	**$\bar{2}$·84464**		**1·15536**	**0·00106**	009	**$\bar{1}$·99894**	**86°**
10′	$\bar{2}$·86128		1·13872	$\bar{2}$·86243		1·13757	0·00115	009	$\bar{1}$·99885	50′
20′	$\bar{2}$·87829		1·12171	$\bar{2}$·87953		1·12047	0·00124	010	$\bar{1}$·99876	40′
30′	$\bar{2}$·89464		1·10536	$\bar{2}$·89598		1·10402	0·00134	010	$\bar{1}$·99866	30′
40′	$\bar{2}$·91040		1·08960	$\bar{2}$·91185		1·08815	0·00144	011	$\bar{1}$·99856	20′
50′	$\bar{2}$·92561		1·07439	$\bar{2}$·92716		1·07284	0·00155	011	$\bar{1}$·99845	10′
5°	**$\bar{2}$·94030**		**1·05970**	**$\bar{2}$·94195**		**1·05805**	**0·00166**	011	**$\bar{1}$·99834**	**85°**
10′	$\bar{2}$·95450		1·04550	$\bar{2}$·95627		1·04373	0·00177	011	$\bar{1}$·99823	50′
20′	$\bar{2}$·96825		1·03175	$\bar{2}$·97013		1·02987	0·00188	012	$\bar{1}$·99812	40′
30′	$\bar{2}$·98157		1·01843	$\bar{2}$·98358		1·01642	0·00200	013	$\bar{1}$·99800	30′
40′	$\bar{2}$·99450		1·00550	$\bar{2}$·99662		1·00338	0·00213	012	$\bar{1}$·99787	20′
50′	$\bar{1}$·00704		0·99296	$\bar{1}$·00930		0·99070	0·00225	014	$\bar{1}$·99775	10′
6°	**$\bar{1}$·01923**		**0·98077**	**$\bar{1}$·02162**		**0·97838**	**0·00239**	013	**$\bar{1}$·99761**	**84°**
10′	$\bar{1}$·03109		0·96891	$\bar{1}$·03361		0·96639	0·00252	014	$\bar{1}$·99748	50′
20′	$\bar{1}$·04262		0·95738	$\bar{1}$·04528		0·95472	0·00266	014	$\bar{1}$·99734	40′
30′	$\bar{1}$·05386		0·94614	$\bar{1}$·05666		0·94334	0·00280	015	$\bar{1}$·99720	30′
40′	$\bar{1}$·06481		0·93519	$\bar{1}$·06775		0·93225	0·00295	015	$\bar{1}$·99705	20′
50′	$\bar{1}$·07548		0·92452	$\bar{1}$·07858		0·92142	0·00310	015	$\bar{1}$·99690	10′
7°	**$\bar{1}$·08589**		**0·91411**	**$\bar{1}$·08914**		**0·91086**	**0·00325**	016	**$\bar{1}$·99675**	**83°**
10′	$\bar{1}$·09606		0·90394	$\bar{1}$·09947		0·90053	0·00341	016	$\bar{1}$·99659	50′
20′	$\bar{1}$·10599	993	0·89401	$\bar{1}$·10956		0·89044	0·00357	016	$\bar{1}$·99643	40′
30′	$\bar{1}$·11570	971	0·88430	$\bar{1}$·11943	987	0·88057	0·00373	017	$\bar{1}$·99627	30′
40′	$\bar{1}$·12519	949	0·87481	$\bar{1}$·12909	966	0·87091	0·00390	017	$\bar{1}$·99610	20′
50′	$\bar{1}$·13447	928	0·86553	$\bar{1}$·13854	945	0·86146	0·00407	018	$\bar{1}$·99593	10′
8°	**$\bar{1}$·14356**	909	**0·85644**	**$\bar{1}$·14780**	926	**0·85220**	**0·00425**	018	**$\bar{1}$·99575**	**82°**
10′	$\bar{1}$·15245	**889**	0·84755	$\bar{1}$·15688	**908**	0·84312	0·00443	018	$\bar{1}$·99557	50′
20′	$\bar{1}$·16116	871	0·83884	$\bar{1}$·16577	889	0·83423	0·00461	019	$\bar{1}$·99539	40′
30′	$\bar{1}$·16970	854	0·83030	$\bar{1}$·17450	873	0·82550	0·00480	019	$\bar{1}$·99520	30′
40′	$\bar{1}$·17807	837	0·82193	$\bar{1}$·18306	856	0·81694	0·00499	019	$\bar{1}$·99501	20′
50′	$\bar{1}$·18628	821	0·81372	$\bar{1}$·19146	840	0·80854	0·00518	020	$\bar{1}$·99482	10′
9°	**$\bar{1}$·19433**	805	**0·80567**	**$\bar{1}$·19971**	825	**0·80029**	**0·00538**	020	**$\bar{1}$·99462**	**81°**
10′	$\bar{1}$·20223	**790**	0·79777	$\bar{1}$·20782	**811**	0·79218	0·00558	021	$\bar{1}$·99442	50′
20′	$\bar{1}$·20999	776	0·79001	$\bar{1}$·21578	796	0·78422	0·00579	021	$\bar{1}$·99421	40′
30′	$\bar{1}$·21761	762	0·78239	$\bar{1}$·22361	783	0·77639	0·00600	021	$\bar{1}$·99400	30′
40′	$\bar{1}$·22509	748	0·77491	$\bar{1}$·23130	769	0·76870	0·00621	022	$\bar{1}$·99379	20′
50′	$\bar{1}$·23244	735	0·76756	$\bar{1}$·23887	757	0·76113	0·00643	022	$\bar{1}$·99357	10′
10°	**$\bar{1}$·23967**	723	**0·76033**	**$\bar{1}$·24632**	745	**0·75368**	**0·00665**		**$\bar{1}$·99335**	**80°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

80°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
10°	·17365	286	5·75877		·17633	300	5·67128		1·01543	052	·98481	51	**80°**
10′	·17651	286	5·66533		·17933	300	5·57638		1·01595	054	·98430	52	50′
20′	·17937	287	5·57493		·18233	301	5·48451		1·01649	054	·98378	53	40′
30′	·18224	285	5·48740		·18534	301	5·39552		1·01703	055	·98325	53	30′
40′	·18509	286	5·40263		·18835	301	5·30928		1·01758	057	·98272	54	20′
50′	·18795	286	5·32049		·19136	302	5·22566		1·01815	057	·98218	55	10′
11°	·19081	285	5·24084		·19438	302	5·14455		1·01872	058	·98163	56	**79°**
10′	·19366	286	5·16359		·19740	302	5·06584		1·01930	059	·98107	57	50′
20′	·19652	285	5·08863		·20042	303	4·98940		1·01989	060	·98050	58	40′
30′	·19937	285	5·01585		·20345	303	4·91516		1·02049	061	·97992	58	30′
40′	·20222	285	4·94517		·20648	304	4·84300		1·02110	061	·97934	59	20′
50′	·20507	284	4·87649		·20952	304	4·77286		1·02171	063	·97875	60	10′
12°	·20791	285	4·80973		·21256	304	4·70463		1·02234	064	·97815	61	**78°**
10′	·21076	284	4·74482		·21560	304	4·63825		1·02298	064	·97754	62	50′
20′	·21360	284	4·68167		·21864	305	4·57363		1·02362	066	·97692	62	40′
30′	·21644	284	4·62023		·22169	306	4·51071		1·02428	066	·97630	64	30′
40′	·21928	284	4·56041		·22475	306	4·44942		1·02494	068	·97566	64	20′
50′	·22212	283	4·50216		·22781	306	4·38969		1·02562	068	·97502	65	10′
13°	·22495	283	4·44541		·23087	306	4·33148		1·02630	070	·97437	66	**77°**
10′	·22778	284	4·39012		·23393	307	4·27471		1·02700	070	·97371	67	50′
20′	·23062	283	4·33622		·23700	308	4·21933		1·02770	071	·97304	67	40′
30′	·23345	282	4·28366		·24008	308	4·16530		1·02841	073	·97237	68	30′
40′	·23627	283	4·23239		·24316	308	4·11256		1·02914	073	·97169	69	20′
50′	·23910	282	4·18238		·24624	309	4·06107		1·02987	074	·97100	70	10′
14°	·24192	282	4·13357		·24933	309	4·01078		1·03061	076	·97030	71	**76°**
10′	·24474	282	4·08591		·25242	310	3·96165		1·03137	076	·96959	72	50′
20′	·24756	282	4·03938		·25552	310	3·91364		1·03213	077	·96887	72	40′
30′	·25038	282	3·99393		·25862	310	3·86671		1·03290	078	·96815	73	30′
40′	·25320	281	3·94952		·26172	311	3·82083		1·03368	079	·96742	75	20′
50′	·25601	281	3·90613		·26483	312	3·77595		1·03447	081	96667	74	10′
15°	·25882	281	3·86370		·26795	312	3·73205		1·03528	081	·96593	76	**75°**
10′	·26163	280	3·82223		·27107	312	3·68909		1·03609	082	·96517	77	50′
20′	·26443	281	3·78166		·27419	313	3·64705		1·03691	083	·96440	77	40′
30′	·26724	280	3·74198		·27732	314	3·60588		1·03774	084	·96363	78	30′
40′	·27004	280	3·70315		·28046	314	3·56557		1·03858	086	·96285	79	20′
50′	·27284	280	3·66515		·28360	315	3·52609		1·03944	086	·96206	80	10′
16°	·27564	279	3·62796		·28675	315	3·48741		1·04030	087	·96126	80	**74°**
10′	·27843	280	3·59154		·28990	315	3·44951		1·04117	089	·96046	82	50′
20′	·28123	279	3·55587		·29305	316	3·41236		1·04206	089	·95964	82	40′
30′	·28402	278	3·52094		·29621	317	3·37594		1·04295	090	·95882	83	30′
40′	·28680	279	3·48671		·29938	317	3·34023		1·04385	092	·95799	84	20′
50′	·28959	278	3·45317		·30255	318	3·30521		1·04477	092	·95715	85	10′
17°	·29237		3·42030		·30573		3·27085		1·04569		·95630	85	**73°**
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

73°

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
10°	**ī·23967**	**710**	**0·76033**	**ī·24632**	**733**	**0·75368**	**0·00665**	**022**	**ī·99335**	**80°**
10′	ī·24677	699	0·75323	ī·25365	721	0·74635	0·00687	023	ī·99313	50′
20′	ī·25376	687	0·74624	ī·26086	711	0·73914	0·00710	023	ī·99290	40′
30′	ī·26063	676	0·73937	ī·26797	699	0·73203	0·00733	024	ī·99267	30′
40′	ī·26739	666	0·73261	ī·27496	690	0·72504	0·00757	024	ī·99243	20′
50′	ī·27405	655	0·72595	ī·28186	679	0·71814	0·00781	024	ī·99219	10′
11°	**ī·28060**	**645**	**0·71940**	**ī·28865**	**670**	**0·71135**	**0·00805**	**025**	**ī·99195**	**79°**
10′	ī·28705	635	0·71295	ī·29535	660	0·70465	0·00830	025	ī·99170	50′
20′	ī·29340	625	0·70660	ī·30195	651	0·69805	0·00855	026	ī·99145	40′
30′	ī·29966	616	0·70034	ī·30846	643	0·69154	0·00881	026	ī·99119	30′
40′	ī·30582	607	0·69418	ī·31489	633	0·68511	0·00907	026	ī·99093	20′
50′	ī·31189	599	0·68811	ī·32122	625	0·67878	0·00933	027	ī·99067	10′
12°	**ī·31788**	**590**	**0·68212**	**ī·32747**	**618**	**0·67253**	**0·00960**	**027**	**ī·99040**	**78°**
10′	ī·32378	582	0·67622	ī·33365	609	0·66635	0·00987	027	ī·99013	50′
20′	ī·32960	574	0·67040	ī·33974	601	0·66026	0·01014	028	ī·98986	40′
30′	ī·33534	566	0·66466	ī·34576	595	0·65424	0·01042	028	ī·98958	30′
40′	ī·34100	558	0·65900	ī·35170	587	0·64830	0·01070	029	ī·98930	20′
50′	ī·34658	551	0·65342	ī·35757	579	0·64243	0·01099	029	ī·98901	10′
13°	**ī·35209**	**543**	**0·64791**	**ī·36336**	**573**	**0·63664**	**0·01128**	**029**	**ī·98872**	**77°**
10′	ī·35752	537	0·64248	ī·36909	567	0·63091	0·01157	030	ī·98843	50′
20′	ī·36289	529	0·63711	ī·37476	559	0·62524	0·01187	030	ī·98813	40′
30′	ī·36819	522	0·63181	ī·38035	554	0·61965	0·01217	030	ī·98783	30′
40′	ī·37341	517	0·62659	ī·38589	547	0·61411	0·01247	031	ī·98753	20′
50′	ī·37858	510	0·62142	ī·39136	541	0·60864	0·01278	032	ī·98722	10′
14°	**ī·38368**	**503**	**0·61632**	**ī·39677**	**535**	**0·60323**	**0·01310**	**031**	**ī·98690**	**76°**
10′	ī·38871	498	0·61129	ī·40212	530	0·59788	0·01341	032	ī·98659	50′
20′	ī·39369	491	0·60631	ī·40742	524	0·59258	0·01373	033	ī·98627	40′
30′	ī·39860	486	0·60140	ī·41266	518	0·58734	0·01406	033	ī·98594	30′
40′	ī·40346	479	0·59654	ī·41784	513	0·58216	0·01439	033	ī·98561	20′
50′	ī·40825	475	0·59175	ī·42297	508	0·57703	0·01472	034	ī·98528	10′
15°	**ī·41300**	**468**	**0·58700**	**ī·42805**	**503**	**0·57195**	**0·01506**	**034**	**ī·98494**	**75°**
10′	ī·41768	464	0·58232	ī·43308	498	0·56692	0·01540	034	ī·98460	50′
20′	ī·42232	458	0·57768	ī·43806	493	0·56194	0·01574	035	ī·98426	40′
30′	ī·42690	453	0·57310	ī·44299	488	0·55701	0·01609	035	ī·98391	30′
40′	ī·43143	448	0·56857	ī·44787	484	0·55213	0·01644	036	ī·98356	20′
50′	ī·43591	443	0·56409	ī·45271	479	0·54729	0·01680	036	ī·98320	10′
16°	**ī·44034**	**438**	**0·55966**	**ī·45750**	**474**	**0·54250**	**0·01716**	**036**	**ī·98284**	**74°**
10′	ī·44472	433	0·55528	ī·46224	470	0·53776	0·01752	037	ī·98248	50′
20′	ī·44905	429	0·55095	ī·46694	466	0·53306	0·01789	037	ī·98211	40′
30′	ī·45334	424	0·54666	ī·47160	462	0·52840	0·01826	038	ī·98174	30′
40′	ī·45758	420	0·54242	ī·47622	458	0·52378	0·01864	038	ī·98136	20′
50′	ī·46178	416	0·53822	ī·48080	454	0·51920	0·01902	038	ī·98098	10′
17°	**ī·46594**		**0·53406**	**ī·48534**		**0·51466**	**0·01940**		**ī·98060**	**73°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
17°	·29237	278	3·42030		·30573	318	3·27085		1·04569	094	·95630	085	**73°**
10′	·29515	278	3·38808		·30891	319	3·23714		1·04663	094	·95545	086	50′
20′	·29793	278	3·35649		·31210	320	3·20406		1·04757	096	·95459	087	40′
30′	·30071	277	3·32551		·31530	320	3·17159		1·04853	097	·95372	088	30′
40′	·30348	277	3·29512		·31850	321	3·13972		1·04950	097	·95284	089	20′
50′	·30625	277	3·26531		·32171	321	3·10842		1·05047	099	·95195	089	10′
18°	·30902	276	3·23607		·32492	322	3·07768		1·05146	100	·95106	091	**72°**
10′	·31178	276	3·20737		·32814	322	3·04749		1·05246	101	·95015	091	50′
20′	·31454	276	3·17920		·33136	324	3·01783		1·05347	102	·94924	092	40′
30′	·31730	276	3·15155		·33460	323	2·98869		1·05449	103	·94832	092	30′
40′	·32006	276	3·12440		·33783	325	2·96004		1·05552	105	·94740	094	20′
50′	·32282	275	3·09774		·34108	325	2·93189		1·05657	105	·94646	094	10′
19°	·32557	275	3·07155		·34433	325	2·90421		1·05762	107	·94552	095	**71°**
10′	·32832	274	3·04584		·34758	327	2·87700		1·05869	107	·94457	096	50′
20′	·33106	275	3·02057		·35085	327	2·85023		1·05976	109	·94361	097	40′
30′	·33381	274	3·99574		·35412	328	2·82391		1·06085	110	·94264	097	30′
40′	·33655	274	2·97135		·35740	328	2·79802		1·06195	111	·94167	099	20′
50′	·33929	273	2·94737		·36068	329	2·77254		1·06306	112	·94068	099	10′
20°	·34202	273	2·92380		·36397	330	2·74748		1·06418	113	·93969	100	**70°**
10′	·34475	273	2·90063		·36727	330	2·72281		1·06531	114	·93869	100	50′
20′	·34748	273	2·87785		·37057	331	2·69853		1·06645	116	·93769	102	40′
30′	·35021	272	2·85545		·37388	332	2·67462		1·06761	117	·93667	102	30′
40′	·35293	272	2·83342		·37720	333	2·65109		1·06878	117	·93565	103	20′
50′	·35565	272	2·81175		·38053	333	2·62791		1·06995	120	·93462	104	10′
21°	·35837	271	2·79043		·38386	335	2·60509		1·07115	120	·93358	105	**69°**
10′	·36108	271	2·76945		·38721	334	2·58261		1·07235	121	·93253	105	50′
20′	·36379	271	2·74881		·39055	336	2·56046		1·07356	123	·93148	106	40′
30′	·36650	271	2·72850		·39391	336	2·53865		1·07479	123	·93042	107	30′
40′	·36921	270	2·70851		·39727	338	2·51715		1·07602	125	·92935	108	20′
50′	·37191	270	2·68884		·40065	338	2·49597		1·07727	126	·92827	109	10′
22°	·37461	269	2·66947		·40403	338	2·47509		1·07853	128	·92718	109	**68°**
10′	·37730	269	2·65040		·40741	340	2·45451		1·07981	128	·92609	110	50′
20′	·37999	269	2·63162		·41081	340	2·43422		1·08109	130	·92499	111	40′
30′	·38268	269	2·61313		·41421	342	2·41421		1·08239	131	·92388	112	30′
40′	·38537	268	2·59491		·41763	342	2·39449		1·08370	133	·92276	112	20′
50′	·38805	268	2·57698		·42105	342	2·37504		1·08503	133	·92164	114	10′
23°	·39073	268	2·55930		·42447	344	2·35585		1·08636	135	·92050	114	**67°**
10′	·39341	267	2·54190		·42791	345	2·33693		1·08771	136	·91936	114	50′
20′	·39608	267	2·52474		·43136	345	2·31826		1·08907	137	·91822	116	40′
30′	·39875	267	2·50784		·43481	347	2·29984		1·09044	139	·91706	116	30′
40′	·40142	266	2·49119		·43828	347	2·28167		1·09183	140	·91590	118	20′
50′	·40408	266	2·47477		·44175	348	2·26374		1·09323	141	·91472	117	10′
24°	·40674		2·45859		·44523		2·24604		1·09464		·91355		**66°**
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

66°

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
17°	**ī·46594**	**401**	**0·53406**	**ī·48534**	**450**	**0·51466**	**0·01940**	**039**	**ī·98060**	**73°**
10′	ī·47005	406	0·52995	ī·48984	446	0·51016	0·01979	039	ī·98021	50′
20′	ī·47411	403	0·52589	ī·49430	442	0·50570	0·02018	040	ī·97982	40′
30′	ī·47814	399	0·52186	ī·49872	439	0·50128	0·02058	040	ī·97942	30′
40′	ī·48213	394	0·51787	ī·50311	435	0·49689	0·02098	041	ī·97902	20′
50′	ī·48607	391	0·51393	ī·50746	432	0·49254	0·02139	040	ī·97861	10′
18°	**ī·48998**	**387**	**0·51002**	**ī·51178**	**428**	**0·48822**	**0·02179**	**042**	**ī·97821**	**72°**
10′	ī·49385	383	0·50615	ī·51606	425	0·48394	0·02221	041	ī·97779	50′
20′	ī·49768	380	0·50232	ī·52031	421	0·47969	0·02262	042	ī·97738	40′
30′	ī·50148	375	0·49852	ī·52452	418	0·47548	0·02304	043	ī·97696	30′
40′	ī·50523	373	0·49477	ī·52870	415	0·47130	0·02347	043	ī·97653	20′
50′	ī·50896	368	0·49104	ī·53285	412	0·46715	0·02390	043	ī·97610	10′
19°	**ī·51264**	**365**	**0·48736**	**ī·53697**	**409**	**0·46303**	**0·02433**	**044**	**ī·97567**	**71°**
10′	ī·51629	362	0·48371	ī·54106	406	0·45894	0·02477	044	ī·97523	50′
20′	ī·51991	359	0·48009	ī·54512	403	0·45488	0·02521	044	ī·97479	40′
30′	ī·52350	355	0·47650	ī·54915	400	0·45085	0·02565	045	ī·97435	30′
40′	ī·52705	352	0·47295	ī·55315	397	0·44685	0·02610	046	ī·97390	20′
50′	ī·53057	348	0·46944	ī·55712	395	0·44288	0·02656	045	ī·97344	10′
20°	**ī·53405**	**346**	**0·46595**	**ī·56107**	**391**	**0·43893**	**0·02701**	**047**	**ī·97299**	**70°**
10′	ī·53751	342	0·46249	ī·56498	389	0·43502	0·02748	046	ī·97252	50′
20′	ī·54093	340	0·45907	ī·56887	387	0·43113	0·02794	047	ī·97206	40′
30′	ī·54433	336	0·45567	ī·57274	384	0·42726	0·02841	048	ī·97159	30′
40′	ī·54769	333	0·45231	ī·57658	381	0·42342	0·02889	048	ī·97111	20′
50′	ī·55102	331	0·44898	ī·58039	379	0·41961	0·02937	048	ī·97063	10′
21°	**ī·55433**	**328**	**0·44567**	**ī·58418**	**376**	**0·41582**	**0·02985**	**049**	**ī·97015**	**69°**
10′	ī·55761	324	0·44239	ī·58794	374	0·41206	0·03034	049	ī·96966	50′
20′	ī·56085	323	0·43915	ī·59168	372	0·40832	0·03083	049	ī·96917	40′
30′	ī·56408	319	0·43592	ī·59540	369	0·40460	0·03132	050	ī·96868	30′
40′	ī·56727	317	0·43273	ī·59909	367	0·40091	0·03182	051	ī·96818	20′
50′	ī·57044	314	0·42956	ī·60276	365	0·39724	0·03233	050	ī·96767	10′
22°	**ī·57358**	**311**	**0·42642**	**ī·60641**	**363**	**0·39359**	**0·03283**	**052**	**ī·96717**	**68°**
10′	ī·57669	309	0·42331	ī·61004	360	0·38996	0·03335	051	ī·96665	50′
20′	ī·57978	306	0·42022	ī·61364	358	0·38636	0·03386	052	ī·96614	40′
30′	ī·58284	304	0·41716	ī·61722	357	0·38278	0·03438	053	ī·96562	30′
40′	ī·58588	301	0·41412	ī·62079	354	0·37921	0·03491	053	ī·96509	20′
50′	ī·58889	299	0·41111	ī·62433	352	0·37567	0·03544	053	ī·96456	10′
23°	**ī·59188**	**296**	**0·40812**	**ī·62785**	**350**	**0·37215**	**0·03597**	**054**	**ī·96403**	**67°**
10′	ī·59484	294	0·40516	ī·63135	349	0·36865	0·03651	055	ī·96349	50′
20′	ī·59778	292	0·40222	ī·63484	346	0·36516	0·03706	054	ī·96294	40′
30′	ī·60070	289	0·39930	ī·63830	345	0·36170	0·03760	055	ī·96240	30′
40′	ī·60359	287	0·39641	ī·64175	342	0·35825	0·03815	056	ī·96185	20′
50′	ī·60646	285	0·39354	ī·64517	341	0·35483	0·03871	056	ī·96129	10′
24°	**ī·60931**		**0·39069**	**ī·64858**		**0·35142**	**0·03927**		**ī·96073**	**66°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
24°	·40674	265	2·45859		·44523	349	2·24604		1·09464	142	·91355	119	66°
10′	·40939	265	2·44264		·44872	350	2·22857		1·09606	144	·91236	120	50′
20′	·41204	265	2·42692		·45222	351	2·21132		1·09750	145	·91116	120	40′
30′	·41469	265	2·41142		·45573	351	2·19430		1·09895	146	·90996	121	30′
40′	·41734	264	2·39614		·45924	353	2·17749		1·10041	148	·90875	122	20′
50′	·41998	264	2·38107		·46277	354	2·16090		1·10189	149	·90753	122	10′
25°	·42262	263	2·36620		·46631	354	2·14451		1·10338	150	·90631	124	65°
10′	·42525	263	2·35154		·46985	356	2·12832		1·10488	152	·90507	124	50′
20′	·42788	263	2·33708		·47341	357	2·11233		1·10640	153	·90383	124	40′
30′	·43051	262	2·32282		·47698	357	2·09654		1·10793	154	·90259	126	30′
40′	·43313	262	2·30875		·48055	359	2·08094		1·10947	156	·90133	126	20′
50′	·43575	262	2·29487		·48414	359	2·06553		1·11103	157	·90007	128	10′
26°	·43837	261	2·28117		·48773	361	2·05030		1·11260	159	·89879	127	64°
10′	·44098	261	2·26766		·49134	361	2·03526		1·11419	160	·89752	129	50′
20′	·44359	261	2·25432		·49495	363	2·02039		1·11579	161	·89623	130	40′
30′	·44620	260	2·24116		·49858	364	2·00569		1·11740	163	·89493	130	30′
40′	·44880	260	2·22817		·50222	365	1·99116		1·11903	164	·89363	131	20′
50′	·45140	259	2·21535		·50587	366	1·97681		1·12067	166	·89232	131	10′
27°	·45399	259	2·20269		·50953	367	1·96261		1·12233	167	·89101	133	63°
10′	·45658	259	2·19019		·51320	368	1·94858		1·12400	168	·88968	133	50′
20′	·45917	258	2·17786		·51688	369	1·93470		1·12568	170	·88835	134	40′
30′	·46175	258	2·16568		·52057	370	1·92098		1·12738	172	·88701	135	30′
40′	·46433	257	2·15366		·52427	371	1·90741		1·12910	173	·88566	135	20′
50′	·46690	257	2·14178		·52798	373	1·89400		1·13083	174	·88431	136	10′
28°	·46947	257	2·13005		·53171	374	1·88073		1·13257	176	·88295	137	62°
10′	·47204	256	2·11847		·53545	375	1·86760		1·13433	177	·88158	138	50′
20′	·47460	256	2·10704		·53920	376	1·85462		1·13610	179	·88020	138	40′
30′	·47716	255	2·09574		·54296	377	1·84177		1·13789	181	·87882	139	30′
40′	·47971	255	2·08458		·54673	378	1·82906		1·13970	182	·87743	140	20′
50′	·48226	255	2·07356		·55051	380	1·81649		1·14152	183	·87603	141	10′
29°	·48481	254	2·06267		·55431	381	1·80405		1·14335	186	·87462	141	61°
10′	·48735	254	2·05191		·55812	382	1·79174		1·14521	186	·87321	143	50′
20′	·48989	253	2·04128		·56194	383	1·77955		1·14707	189	·87178	142	40′
30′	·49242	253	2·03077		·56577	385	1·76749		1·14896	189	·87036	144	30′
40′	·49495	253	2·02039		·56962	386	1·75556		1·15085	192	·86892	144	20′
50′	·49748	253	2·01014		·57348	387	1·74375		1·15277	193	·86748	145	10′
30°	·50000	252	2·00000		·57735	389	1·73205		1·15470	195	·86603	146	60°
10′	·50252	251	1·98998	990	·58124	389	1·72047		1·15665	196	·86457	147	50′
20′	·50503	251	1·98008	979	·58513	392	1·70901		1·15861	198	·86310	147	40′
30′	·50754	250	1·97029	967	·58905	392	1·69766		1·16059	200	·86163	148	30′
40′	·51004	250	1·96062	956	·59297	394	1·68643		1·16259	201	·86015	149	20′
50′	·51254	250	1·95106	946	·59691	395	1·67530		1·16460	203	·85866	149	10′
31°	·51504		1·94160		·60086		1·66428		1·16663		·85717		59°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

59°

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
24°	ī·60931	283	0·39069	ī·64858	339	0·35142	0·03927	056	ī·96073	66°
10′	ī·61214	280	0·38786	ī·65197	338	0·34803	0·03983	057	ī·96017	50′
20′	ī·61494	279	0·38506	ī·65535	335	0·34465	0·04040	058	ī·95960	40′
30′	ī·61773	276	0·38227	ī·65870	334	0·34130	0·04098	057	ī·95902	30′
40′	ī·62049	274	0·37951	ī·66204	333	0·33796	0·04156	059	ī·95845	20′
50′	ī·62323	272	0·37677	ī·66537	330	0·33463	0·04214	058	ī·95786	10′
25°	ī·62595	270	0·37405	ī·66867	329	0·33133	0·04272	060	ī·95728	65°
10′	ī·62865	268	0·37135	ī·67196	328	0·32804	0·04332	059	ī·95668	50′
20′	ī·63133	265	0·36867	ī·67524	326	0·32476	0·04391	060	ī·95609	40′
30′	ī·63398	264	0·36602	ī·67850	324	0·32150	0·04451	061	ī·95549	30′
40′	ī·63662	262	0·36338	ī·68174	323	0·31826	0·04512	061	ī·95488	20′
50′	ī·63924	260	0·36076	ī·68497	321	0·31503	0·04573	061	ī·95427	10′
26°	ī·64184	258	0·35816	ī·68818	320	0·31182	0·04634	062	ī·95366	64°
10′	ī·64442	256	0·35558	ī·69138	319	0·30862	0·04696	062	ī·95304	50′
20′	ī·64698	255	0·35302	ī·69457	317	0·30543	0·04758	063	ī·95242	40′
30′	ī·64953	252	0·35047	ī·69774	315	0·30226	0·04821	063	ī·95179	30′
40′	ī·65205	251	0·34795	ī·70089	315	0·29911	0·04884	064	ī·95116	20′
50′	ī·65456	249	0·34544	ī·70404	313	0·29596	0·04948	064	ī·95052	10′
27°	ī·65705	247	0·34295	ī·70717	311	0·29283	0·05012	065	ī·94988	63°
10′	ī·65952	245	0·34048	ī·71028	311	0·28972	0·05077	065	ī·94923	50′
20′	ī·66197	244	0·33803	ī·71339	309	0·28661	0·05142	065	ī·94858	40′
30′	ī·66441	241	0·33559	ī·71648	307	0·28352	0·05207	066	ī·94793	30′
40′	ī·66682	241	0·33318	ī·71955	307	0·28045	0·05273	067	ī·94727	20′
50′	ī·66923	238	0·33078	ī·72262	305	0·27738	0·05340	067	ī·94660	10′
28°	ī·67161	237	0·32839	ī·72567	305	0·27433	0·05407	067	ī·94593	62°
10′	ī·67398	235	0·32602	ī·72872	303	0·27128	0·05474	068	ī·94526	50′
20′	ī·67633	233	0·32367	ī·73175	301	0·26825	0·05542	068	ī·94458	40′
30′	ī·67866	232	0·32134	ī·73476	301	0·26524	0·05610	069	ī·94390	30′
40′	ī·68098	230	0·31902	ī·73777	300	0·26223	0·05679	069	ī·94321	20′
50′	ī·68328	229	0·31672	ī·74077	298	0·25923	0·05748	070	ī·94252	10′
29°	ī·68557	227	0·31443	ī·74375	298	0·25625	0·05818	070	ī·94182	61°
10′	ī·68784	226	0·31216	ī·74673	296	0·25327	0·05888	071	ī·94112	50′
20′	ī·69010	224	0·30990	ī·74969	295	0·25031	0·05959	071	ī·94041	40′
30′	ī·69234	222	0·30766	ī·75264	294	0·24736	0·06030	072	ī·93970	30′
40′	ī·69456	221	0·30544	ī·75558	294	0·24442	0·06102	072	ī·93898	20′
50′	ī·69677	220	0·30323	ī·75852	292	0·24148	0·06174	073	ī·93826	10′
30°	ī·69897	218	0·30103	ī·76144	291	0·23856	0·06247	073	ī·93753	60°
10′	ī·70115	217	0·29885	ī·76435	290	0·23565	0·06320	074	ī·93680	50′
20′	ī·70332	215	0·29668	ī·76726	290	0·23275	0·06394	074	ī·93606	40′
30′	ī·70547	214	0·29453	ī·77015	288	0·22985	0·06468	075	ī·93532	30′
40′	ī·70761	212	0·29239	ī·77303	288	0·22697	0·06543	075	ī·93457	20′
50′	ī·70973	211	0·29027	ī·77591	286	0·22409	0·06618	075	ī·93382	10′
31°	ī·71184		0·28816	ī·77877		0·22123	0·06693		ī·93307	59°
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

59°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
31°	·51504	249	1·94160	934	·60086	397	1·66428		1·16663	205	·85717	150	60°
10′	·51753	249	1·93226	924	·60483	398	1·65337		1·16868	207	·85567	151	50′
20′	·52002	248	1·92302	914	·60881	399	1·64256		1·17075	208	·85416	152	40′
30′	·52250	248	1·91388	903	·61280	401	1·63185		1·17283	210	·85264	152	30′
40′	·52498	247	1·90485	894	·61681	402	1·62125		1·17493	211	·85112	153	20′
50′	·52745	247	1·89591	883	·62083	404	1·61074		1·17704	214	·84959	154	10′
32°	·52992	246	1·88708	874	·62487	405	1·60033		1·17918	215	·84805	155	58°
10′	·53238	246	1·87834	864	·62892	407	1·59002		1·18133	217	·84650	155	50′
20′	·53484	246	1·86970	854	·63299	408	1·57981		1·18350	219	·84495	156	40′
30′	·53730	245	1·86116	845	·63707	410	1·56969		1·18569	221	·84339	157	30′
40′	·53975	245	1·85271	836	·64117	411	1·55966		1·18790	222	·84182	157	20′
50′	·54220	244	1·84435	827	·64528	413	1·54972	994	1·19012	224	·84025	158	10′
33°	·54464	244	1·83608	818	·64941	414	1·53987	985	1·19236	227	·83867	159	57°
10′	·54708	243	1·82790	809	·65355	416	1·53010	977	1·19463	228	·83708	159	50′
20′	·54951	243	1·81981	801	·65771	418	1·52043	967	1·19691	229	·83549	160	40′
30′	·55194	242	1·81180	792	·66189	419	1·51084	959	1·19920	232	·83389	161	30′
40′	·55436	242	1·80388	784	·66608	420	1·50133	951	1·20152	234	·83228	162	20′
50′	·55678	241	1·79604	775	·67028	423	1·49190	943	1·20386	236	·83066	162	10′
34°	·55919	241	1·78829	767	·67451	424	1·48256	934	1·20622	237	·82904	163	56°
10′	·56160	241	1·78062	759	·67875	426	1·47330	926	1·20859	240	·82741	164	50′
20′	·56401	240	1·77303	751	·68301	427	1·46411	919	1·21099	242	·82577	164	40′
30′	·56641	239	1·76552	744	·68728	429	1·45501	910	1·21341	243	·82413	165	30′
40′	·56880	239	1·75808	735	·69157	431	1·44598	903	1·21584	246	·82248	166	20′
50′	·57119	239	1·75073	728	·69588	433	1·43703	895	1·21830	247	·82082	167	10′
35°	·57358	238	1·74345	721	·70021	434	1·42815	888	1·22077	250	·81915	167	55°
10′	·57596	237	1·73624	713	·70455	436	1·41934	881	1·22327	252	·81748	168	50′
20′	·57833	237	1·72911	706	·70891	438	1·41061	873	1·22579	254	·81580	168	40′
30′	·58070	237	1·72205	699	·71329	440	1·40195	866	1·22833	256	·81412	170	30′
40′	·58307	236	1·71506	691	·71769	442	1·39336	859	1·23089	258	·81242	170	20′
50′	·58543	236	1·70815	685	·72211	443	1·38484	852	1·23347	260	·81072	170	10′
36°	·58779	235	1·70130	678	·72654	446	1·37638	846	1·23607	262	·80902	172	54°
10′	·59014	234	1·69452	670	·73100	447	1·36800	838	1·23869	265	·80730	172	50′
20′	·59248	234	1·68782	665	·73547	449	1·35968	832	1·24134	266	·80558	172	40′
30′	·59482	234	1·68117	657	·73996	451	1·35142	826	1·24400	269	·80386	174	30′
40′	·59716	233	1·67460	651	·74447	453	1·34323	819	1·24669	271	·80212	174	20′
50′	·59949	233	1·66809	645	·74900	455	1·33511	812	1·24940	274	·80038	174	10′
37°	·60182	232	1·66164	638	·75355	457	1·32704	807	1·25214	275	·79864	176	53°
10′	·60414	231	1·65526	632	·75812	460	1·31904	800	1·25489	278	·79688	176	50′
20′	·60645	231	1·64894	626	·76272	461	1·31110	794	1·25767	280	·79512	177	40′
30′	·60876	231	1·64268	620	·76733	463	1·30323	787	1·26047	283	·79335	177	30′
40′	·61107	230	1·63648	613	·77196	465	1·29541	782	1·26330	285	·79158	178	20′
50′	·61337	229	1·63035	608	·77661	468	1·28764	777	1·26615	287	·78980	179	10′
38°	·61566		1·62427		·78129		1·27994	770	1·26902		·78801		52°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
31°	Ī·71184	209	0·28816	Ī·77877	286	0·22123	0·06693	077	Ī·93307	59°
10′	Ī·71393	209	0·28607	Ī·78163	285	0·21837	0·06770	076	Ī·93230	50′
20′	Ī·71602	207	0·28398	Ī·78448	284	0·21552	0·06846	077	Ī·93154	40′
30′	Ī·71809	205	0·28191	Ī·78732	283	0·21268	0·06923	078	Ī·93077	30′
40′	Ī·72014	204	0·27986	Ī·79015	282	0·20985	0·07001	078	Ī·92999	20′
50′	Ī·72218	203	0·27782	Ī·79297	282	0·20703	0·07079	079	Ī·92921	10′
32°	Ī·72421	201	0·27579	Ī·79579	281	0·20421	0·07158	079	Ī·92842	58°
10′	Ī·72622	201	0·27378	Ī·79860	280	0·20140	0·07237	080	Ī·92763	50′
20′	Ī·72823	199	0·27177	Ī·80140	279	0·19860	0·07317	080	Ī·92683	40′
30′	Ī·73022	197	0·26978	Ī·80419	278	0·19581	0·07397	081	Ī·92603	30′
40′	Ī·73219	197	0·26781	Ī·80697	278	0·19303	0·07478	081	Ī·92522	20′
50′	Ī·73416	195	0·26584	Ī·80975	277	0·19025	0·07559	082	Ī·92441	10′
33°	Ī·73611	194	0·26389	Ī·81252	276	0·18748	0·07641	082	Ī·92359	57°
10′	Ī·73805	192	0·26195	Ī·81528	275	0·18472	0·07723	083	Ī·92277	50′
20′	Ī·73997	192	0·26003	Ī·81803	275	0·18197	0·07806	083	Ī·92194	40′
30′	Ī·74189	190	0·25811	Ī·82078	274	0·17922	0·07889	084	Ī·92111	30′
40′	Ī·74379	189	0·25621	Ī·82352	274	0·17648	0·07973	085	Ī·92027	20′
50′	Ī·74568	188	0·25432	Ī·82626	273	0·17374	0·08058	085	Ī·91942	10′
34°	Ī·74756	187	0·25244	Ī·82899	272	0·17101	0·08143	085	Ī·91857	56°
10′	Ī·74943	185	0·25057	Ī·83171	271	0·16829	0·08228	086	Ī.91772	50′
20′	Ī·75128	185	0·24872	Ī·83442	271	0·16558	0·08314	087	Ī·91686	40′
30′	Ī·75313	183	0·24687	Ī·83713	271	0·16287	0·08401	087	Ī·91599	30′
40′	Ī·75496	182	0·24504	Ī·83984	270	0·16016	0·08488	087	Ī·91512	20′
50′	Ī·75678	181	0·24322	Ī·84254	269	0·15746	0·08575	089	Ī·91425	10′
35°	Ī·75859	180	0·24141	Ī·84523	268	0·15477	0·08664	088	Ī·91336	55°
10′	Ī·76039	179	0·23961	Ī·84791	268	0·15209	0·08752	090	Ī·91248	50′
20′	Ī·76218	177	0·23782	Ī·85059	268	0·14941	0·08842	089	Ī·91158	40′
30′	Ī·76395	177	0·23605	Ī·85327	267	0·14673	0·08931	091	Ī·91069	30′
40′	Ī·76572	175	0·23428	Ī·85594	266	0·14406	0·09022	091	Ī·90978	20′
50′	Ī·76747	175	0·23253	Ī·85860	266	0·14140	0·09113	091	Ī·90887	10′
36°	Ī·76922	173	0·23078	Ī·86126	266	0·13874	0·09204	092	Ī·90796	54°
10′	Ī·77095	173	0·22905	Ī·86392	264	0·13608	0·09296	093	Ī·90704	50′
20′	Ī·77268	171	0·22732	Ī·86656	265	0·13344	0·09389	093	Ī·90611	40′
30′	Ī·77439	170	0·22561	Ī·86921	264	0·13079	0·09482	094	Ī·90518	30′
40′	Ī·77609	169	0·22391	Ī·87185	263	0·12815	0·09576	094	Ī·90424	20′
50′	Ī·77778	168	0·22222	Ī·87448	263	0·12552	0·09670	095	Ī·90330	10′
37°	Ī·77946	167	0·22054	Ī·87711	263	0·12289	0·09765	096	Ī·90235	53°
10′	Ī·78113	167	0·21887	Ī·87974	262	0·12026	0·09861	096	Ī·90139	50′
20′	Ī·78280	165	0·21720	Ī·88236	262	0·11764	0·09957	096	Ī·90043	40′
30′	Ī·78445	164	0·21555	Ī·88498	261	0·11502	0·10053	098	Ī·89947	30′
40′	Ī·78609	163	0·21391	Ī·88759	261	0·11241	0·10151	097	Ī·89849	20′
50′	Ī·78772	162	0·21228	Ī·89020	261	0·10980	0·10248	099	Ī·89752	10′
38°	Ī·78934		0·21066	Ī·89281		0·10719	0·10347		Ī·89653	52°
	log cos	D	log sec	log cotan		log tan	log cosec	D	log sin	

52°

	sine	D	cosecant	D	tangent	D	cotangent	D	secant	D	cosine	D	
38°	·61566	229	1·62427	602	·78129	469	1·27994	764	1·26902	289	·78801	179	52°
10′	·61795	229	1·61825	596	·78598	472	1·27230	759	1·27191	292	·78622	180	50′
20′	·62024	227	1·61229	590	·79070	474	1·26471	754	1·27483	295	·78442	181	40′
30′	·62251	228	1·60639	585	·79544	476	1·25717	748	1·27778	297	·78261	182	30′
40′	·62479	227	1·60054	579	·80020	478	1·24969	742	1·28075	299	·78079	182	20′
50′	·62706	226	1·59475	573	·80498	480	1·24227	737	1·28374	302	·77897	182	10′
39°	·62932	226	1·58902	569	·80978	483	1·23490	732	1·28676	304	·77715	184	51°
10′	·63158	225	1·58333	562	·81461	485	1·22758	727	1·28980	307	·77531	184	50′
20′	·63383	225	1·57771	558	·81946	488	1·22031	721	1·29287	310	·77347	185	40′
30′	·63608	224	1·57213	552	·82434	489	1·21310	717	1·29597	312	·77162	185	30′
40′	·63832	224	1·56661	547	·82923	492	1·20593	711	1·29909	314	·76977	186	20′
50′	·64056	223	1·56114	542	·83415	495	1·19882	707	1·30223	318	·76791	187	10′
40°	·64279	222	1·55572	536	·83910	497	1·19175	701	1·30541	320	·76604	187	50°
10′	·64501	222	1·55036	532	·84407	499	1·18474	697	1·30861	322	·76417	188	50′
20′	·64723	222	1·54504	527	·84906	502	1·17777	692	1·31183	326	·76229	188	40′
30′	·64945	221	1·53977	522	·85408	504	1·17085	687	1·31509	328	·76041	190	30′
40′	·65166	220	1·53455	517	·85912	507	1·16398	683	1·31837	331	·75851	190	20′
50′	·65386	220	1·52938	513	·86419	510	1·15715	678	1·32168	333	·75661	190	10′
41°	·65606	219	1·52425	507	·86929	512	1·15037	674	1·32501	337	·75471	191	49°
10′	·65825	219	1·51918	503	·87441	514	1·14363	669	1·32838	339	·75280	192	50′
20′	·66044	218	1·51415	499	·87955	518	1·13694	665	1·33177	342	·75088	192	40′
30′	·66262	218	1·50916	494	·88473	519	1·13029	660	1·33519	345	·74896	193	30′
40′	·66480	217	1·50422	489	·88992	523	1·12369	656	1·33864	348	·74703	194	20′
50′	·66697	216	1·49933	485	·89515	525	1·11713	652	1·34212	351	·74509	195	10′
42°	·66913	216	1·49448	481	·90040	529	1·11061	647	1·34563	354	·74314	194	48°
10′	·67129	215	1·48967	476	·90569	530	1·10414	644	1·34917	357	·74120	196	50′
20′	·67344	215	1·48491	472	·91099	534	1·09770	639	1·35274	360	·73924	196	40′
30′	·67559	214	1·48019	468	·91633	537	1·09131	635	1·35634	363	·73728	197	30′
40′	·67773	214	1·47551	464	·92170	539	1·08496	632	1·35997	366	·73531	198	20′
50′	·67987	213	1·47087	459	·92709	543	1·07864	627	1·36363	370	·73333	198	10′
43°	·68200	212	1·46628	455	·93252	545	1·07237	624	1·36733	372	·73135	198	47°
10′	·68412	212	1·46173	452	·93797	548	1·06613	619	1·37105	376	·72937	200	50′
20′	·68624	211	1·45721	447	·94345	551	1·05994	616	1·37481	379	·72737	200	40′
30′	·68835	211	1·45274	443	·94896	555	1·05378	612	1·37860	382	·72537	200	30′
40′	·69046	210	1·44831	440	·95451	557	1·04766	608	1·38242	386	·72337	201	20′
50′	·69256	210	1·44391	435	·96008	561	1·04158	605	1·38628	388	·72136	202	10′
44°	·69466	206	1·43956	432	·96569	564	1·03553	601	1·39016	393	·71934	202	46°
10′	·69675	208	1·43524	428	·97133	567	1·02952	597	1·39409	395	·71732	203	50′
20′	·69883	208	1·43096	424	·97700	570	1·02355	594	1·39804	399	·71529	204	40′
30′	·70091	207	1·42672	421	·98270	573	1·01761	591	1·40203	403	·71325	204	30′
40′	·70298	207	1·42251	416	·98843	577	1·01170	587	1·40606	406	·71121	205	20′
50′	·70505	206	1·41835	414	·99420	580	1·00583	583	1·41012	409	·70916	205	10′
45°	·70711		1·41421		1·00000		1·00000		1·41421		·70711		45°
	cosine	D	secant	D	cotangent	D	tangent	D	cosecant	D	sine	D	

45°

	log sin	D	log cosec	log tan	D	log cotan	log sec	D	log cos	
38°	$\bar{1}\cdot78934$	161	0·21066	$\bar{1}\cdot89281$	260	0·10719	0·10347	099	$\bar{1}\cdot89653$	**52°**
10′	$\bar{1}\cdot79095$	161	0·20905	$\bar{1}\cdot89541$	260	0·10459	0·10446	099	$\bar{1}\cdot89554$	50′
20′	$\bar{1}\cdot79256$	159	0·20744	$\bar{1}\cdot89801$	260	0·10199	0·10545	101	$\bar{1}\cdot89455$	40′
30′	$\bar{1}\cdot79415$	158	0·20585	$\bar{1}\cdot90061$	259	0·09939	0·10646	100	$\bar{1}\cdot89354$	30′
40′	$\bar{1}\cdot79573$	158	0·20427	$\bar{1}\cdot90320$	258	0·09680	0·10746	102	$\bar{1}\cdot89254$	20′
50′	$\bar{1}\cdot79731$	156	0·20269	$\bar{1}\cdot90578$	259	0·09422	0·10848	102	$\bar{1}\cdot89152$	10′
39°	$\bar{1}\cdot79887$	**156**	**0·20113**	$\bar{1}\cdot90837$	**258**	**0·09163**	**0·10950**	**102**	$\bar{1}\cdot89050$	**51°**
10′	$\bar{1}\cdot80043$	154	0·19957	$\bar{1}\cdot91095$	258	0·08905	0·11052	104	$\bar{1}\cdot88948$	50′
20′	$\bar{1}\cdot80197$	154	0·19803	$\bar{1}\cdot91353$	257	0·08647	0·11156	103	$\bar{1}\cdot88844$	40′
30′	$\bar{1}\cdot80351$	153	0·19649	$\bar{1}\cdot91610$	258	0·08390	0·11259	105	$\bar{1}\cdot88741$	30′
40′	$\bar{1}\cdot80504$	152	0·19496	$\bar{1}\cdot91868$	257	0·08132	0·11364	105	$\bar{1}\cdot88636$	20′
50′	$\bar{1}\cdot80656$	151	0·19344	$\bar{1}\cdot92125$	256	0·07875	0·11469	106	$\bar{1}\cdot88531$	10′
40°	$\bar{1}\cdot80807$	**150**	**0·19193**	$\bar{1}\cdot92381$	**257**	**0·07619**	**0·11575**	**106**	$\bar{1}\cdot88425$	**50°**
10′	$\bar{1}\cdot80957$	149	0·19043	$\bar{1}\cdot92638$	256	0·07362	0·11681	107	$\bar{1}\cdot88319$	50′
20′	$\bar{1}\cdot81106$	148	0·18894	$\bar{1}\cdot92894$	256	0·07106	0·11788	107	$\bar{1}\cdot88212$	40′
30′	$\bar{1}\cdot81254$	148	0·18746	$\bar{1}\cdot93150$	256	0·06850	0·11895	109	$\bar{1}\cdot88105$	30′
40′	$\bar{1}\cdot81402$	147	0·18598	$\bar{1}\cdot93406$	255	0·06594	0·12004	109	$\bar{1}\cdot87996$	20′
50′	$\bar{1}\cdot81549$	145	0·18451	$\bar{1}\cdot93661$	255	0·06339	0·12113	109	$\bar{1}\cdot87887$	10′
41°	$\bar{1}\cdot81694$	**145**	**0·18306**	$\bar{1}\cdot93916$	**255**	**0·06084**	**0·12222**	**110**	$\bar{1}\cdot87778$	**49°**
10′	$\bar{1}\cdot81839$	144	0·18161	$\bar{1}\cdot94171$	255	0·05829	0·12332	111	$\bar{1}\cdot87668$	50′
20′	$\bar{1}\cdot81983$	143	0·18017	$\bar{1}\cdot94426$	255	0·05574	0·12443	111	$\bar{1}\cdot87557$	40′
30′	$\bar{1}\cdot82126$	143	0·17874	$\bar{1}\cdot94681$	254	0·05319	0·12554	112	$\bar{1}\cdot87446$	30′
40′	$\bar{1}\cdot82269$	141	0·17731	$\bar{1}\cdot94935$	255	0·05065	0·12666	113	$\bar{1}\cdot87334$	20′
50′	$\bar{1}\cdot82410$	141	0·17590	$\bar{1}\cdot95190$	254	0·04810	0·12779	114	$\bar{1}\cdot87221$	10′
42°	$\bar{1}\cdot82551$	**140**	**0·17449**	$\bar{1}\cdot95444$	**254**	**0·04556**	**0·12893**	**114**	$\bar{1}\cdot87107$	**48°**
10′	$\bar{1}\cdot82691$	139	0·17309	$\bar{1}\cdot95698$	254	0·04302	0·13007	114	$\bar{1}\cdot86993$	50′
20′	$\bar{1}\cdot82830$	138	0·17170	$\bar{1}\cdot95952$	253	0·04048	0·13121	116	$\bar{1}\cdot86879$	40′
30′	$\bar{1}\cdot82968$	138	0·17032	$\bar{1}\cdot96205$	254	0·03795	0·13237	116	$\bar{1}\cdot86763$	30′
40′	$\bar{1}\cdot83106$	136	0·16894	$\bar{1}\cdot96459$	253	0·03541	0·13353	117	$\bar{1}\cdot86647$	20′
50′	$\bar{1}\cdot83242$	136	0·16758	$\bar{1}\cdot96712$	254	0·03288	0·13470	117	$\bar{1}\cdot86530$	10′
43°	$\bar{1}\cdot83378$	**135**	**0·16622**	$\bar{1}\cdot96966$	**253**	**0·03034**	**0·13587**	**118**	$\bar{1}\cdot86413$	**47°**
10′	$\bar{1}\cdot83513$	135	0·16487	$\bar{1}\cdot97219$	253	0·02781	0·13705	119	$\bar{1}\cdot86295$	50′
20′	$\bar{1}\cdot83648$	133	0·16352	$\bar{1}\cdot97472$	253	0·02528	0·13824	120	$\bar{1}\cdot86176$	40′
30′	$\bar{1}\cdot83781$	133	0·16219	$\bar{1}\cdot97725$	253	0·02275	0·13944	120	$\bar{1}\cdot86056$	30′
40′	$\bar{1}\cdot83914$	132	0·16086	$\bar{1}\cdot97978$	253	0·02022	0·14064	121	$\bar{1}\cdot85936$	20′
50′	$\bar{1}\cdot84046$	131	0·15954	$\bar{1}\cdot98231$	253	0·01769	0·14185	122	$\bar{1}\cdot85815$	10′
44°	$\bar{1}\cdot84177$	**131**	**0·15823**	$\bar{1}\cdot98484$	**253**	**0·01516**	**0·14307**	**122**	$\bar{1}\cdot85693$	**46°**
10′	$\bar{1}\cdot84308$	129	0·15692	$\bar{1}\cdot98737$	252	0·01263	0·14429	123	$\bar{1}\cdot85571$	50′
20′	$\bar{1}\cdot84437$	129	0·15563	$\bar{1}\cdot98989$	253	0·01011	0·14552	124	$\bar{1}\cdot85448$	40′
30′	$\bar{1}\cdot84566$	128	0·15434	$\bar{1}\cdot99242$	253	0·00758	0·14676	124	$\bar{1}\cdot85324$	30′
40′	$\bar{1}\cdot84694$	128	0·15306	$\bar{1}\cdot99495$	253	0·00505	0·14800	126	$\bar{1}\cdot85200$	20′
50′	$\bar{1}\cdot84822$	127	0·15178	$\bar{1}\cdot99747$	252	0·00253	0·14926	126	$\bar{1}\cdot85074$	10′
45°	$\bar{1}\cdot84949$		**0·15052**	**0·00000**	253	**0·00000**	**0·15052**		$\bar{1}\cdot84949$	**45°**
	log cos	D	log sec	log cotan	D	log tan	log cosec	D	log sin	

45°

Radians	Degrees	Sine	Cosec.	Tangent	Cotan.	Secant	Cosine		
·00000	0	·0000	∞	·0000	∞	1·0000	1·0000	90	1·57080
·01745	1	·0175	57·2986	·0175	57·2899	1·0002	·9998	89	1·55334
·03491	2	·0349	28·6537	·0349	28·6362	1·0006	·9994	88	1·53589
·05236	3	·0523	19·1073	·0524	19·0811	1·0014	·9986	87	1·51844
·06981	4	·0698	14·3356	·0699	14·3006	1·0024	·9976	86	1·50098
·08727	5	·0872	11·4737	·0875	11·4301	1·0038	·9962	85	1·48353
·10472	6	·1045	9·5668	·1051	9·5144	1·0055	·9945	84	1·46608
·12217	7	·1219	8·2055	·1228	8·1443	1·0075	·9925	83	1·44862
·13963	8	·1392	7·1853	·1405	7·1154	1·0098	·9903	82	1·43117
·15708	9	·1564	6·3925	·1584	6·3138	1·0125	·9877	81	1·41372
·17453	10	·1736	5·7588	·1763	5·6713	1·0154	·9848	80	1·39626
·19199	11	·1908	5·2408	·1944	5·1446	1·0187	·9816	79	1·37881
·20944	12	·2079	4·8097	·2126	4·7046	1·0223	·9781	78	1·36136
·22689	13	·2250	4·4454	·2309	4·3315	1·0263	·9744	77	1·34390
·24435	14	·2419	4·1336	·2493	4·0108	1·0306	·9703	76	1·32645
·26180	15	·2588	3·8637	·2679	3·7321	1·0353	·9659	75	1·30900
·27925	16	·2756	3·6280	·2867	3·4874	1·0403	·9613	74	1·29154
·29671	17	·2924	3·4203	·3057	3·2709	1·0457	·9563	73	1·27409
·31416	18	·3090	3·2361	·3249	3·0777	1·0515	·9511	72	1·25664
·33161	19	·3256	3·0716	·3443	2·9042	1·0576	·9455	71	1·23918
·34907	20	·3420	2·9238	·3640	2·7475	1·0642	·9397	70	1·22173
·36652	21	·3584	2·7904	·3839	2·6051	1·0711	·9336	69	1·20428
·38397	22	·3746	2·6695	·4040	2·4751	1·0785	·9272	68	1·18682
·40143	23	·3907	2·5593	·4245	2·3559	1·0864	·9205	67	1·16937
·41888	24	·4067	2·4586	·4452	2·2460	1·0946	·9135	66	1·15192
·43633	25	·4226	2·3662	·4663	2·1445	1·1034	·9063	65	1·13446
·45379	26	·4384	2·2812	·4877	2·0503	1·1126	·8988	64	1·11701
·47124	27	·4540	2·2027	·5095	1·9626	1·1223	·8910	63	1·09956
·48869	28	·4695	2·1301	·5317	1·8807	1·1326	·8829	62	1·08210
·50615	29	·4848	2·0627	·5543	1·8040	1·1434	·8746	61	1·06465
·52360	30	·5000	2·0000	·5774	1·7321	1·1547	·8660	60	1·04720
·54105	31	·5150	1·9416	·6009	1·6643	1·1666	·8572	59	1·02974
·55851	32	·5299	1·8871	·6249	1·6003	1·1792	·8480	58	1·01229
·57596	33	·5446	1·8361	·6494	1·5399	1·1924	·8387	57	·99484
·59341	34	·5592	1·7883	·6745	1·4826	1·2062	·8290	56	·97738
·61087	35	·5736	1·7434	·7002	1·4281	1·2208	·8192	55	·95993
·62832	36	·5878	1·7013	·7265	1·3764	1·2361	·8090	54	·94248
·64577	37	·6018	1·6616	·7536	1·3270	1·2521	·7986	53	·92502
·66323	38	·6157	1·6243	·7813	1·2799	1·2690	·7880	52	·90757
·68068	39	·6293	1·5890	·8098	1·2349	1·2868	·7771	51	·89012
·69813	40	·6428	1·5557	·8391	1·1918	1·3054	·7660	50	·87266
·71559	41	·6561	1·5243	·8693	1·1504	1·3250	·7547	49	·85521
·73304	42	·6691	1·4945	·9004	1·1106	1·3456	·7431	48	·83776
·75049	43	·6820	1·4663	·9325	1·0724	1·3673	·7314	47	·82030
·76794	44	·6947	1·4396	·9657	1·0355	1·3902	·7193	46	·80285
·78540	45	·7071	1·4142	1·0000	1·0000	1·4142	·7071	45	·78540
		Cosine	Secant	Cotan.	Tangent	Cosec.	Sine	Degrees	Radians

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